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Rigorous relativistic equation for quark–antiquark bound states at finite temperature derived from thermal QCD formulated in the coherent-state representation

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Abstract. A rigorous three-dimensional relativistic equation for quark–antiquark bound states at finite temperature is derived from the thermal QCD generating functional which is formulated in the coherent-state representation. The generating functional is derived newly and given a correct path-integral expression. The perturbative expansion of the generating functional is specifically given by means of the stationary-phase method. Especially, the interaction kernel in the three-dimensional equation is derived by virtue of the equations of motion satisfied by some quark–antiquark Green functions and given a closed form which is expressed in terms of only a few types of Green functions. This kernel is very suitable to use for exploring the deconfinement of quarks. To demonstrate the applicability of the equation derived, the one-gluon exchange kernel is derived and described in detail.

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1 Introduction

Quantum chromodynamics (QCD), as a strong interaction theory of quarks and gluons, has a distinctive property of asymptotic freedom and infrared slavery which makes the quarks and gluons to be confined in hadrons. It is widely believed that in extreme conditions, i.e., at high temperature and/or high density, the quarks and gluons would be deconfined from hadrons and form a new matter, the quark–gluon plasma (QGP). It is highly expected that the QCD phase transition from hadrons to QGP would take place and would be observed in high energy heavy ion collisions at RHIC [1–3]. Theoretically, to predict the QCD phase transition, many efforts have been made by using different approaches such as the lattice simulation, the effective field theory, the hydrodynamic model etc. [4–14]. According to the prediction of the lattice QCD calculations, the phase transition could occur when the colliding system reaches temperatures 150–170 MeV [2]. Since the quarks and gluons are confined in hadrons, which exist as bound states of quarks and/or gluons, it is obvious that a proper approach of investigating the quark deconfinement is to start from an exact relativistic equation for quark and/or antiquark bound states at finite temperature. This paper is devoted to deriving a rigorous three-dimensional relativistic equation of Dirac–Schrödinger type for quark–antiquark ($q\overline{q}$) bound states at finite temperature. In particular, the interaction kernel in the equation will be given a closed and explicit expression which will be derived by following the procedure described in [15–17]. This constitutes the main purpose of this paper. Clearly, if the interaction kernel and the equation can be calculated by a suitable nonperturbation method, one can exactly determine at which temperature the quark and antiquark will be deconfined from mesons.

In this paper, we intend to derive the aforementioned equation and kernel from the thermal QCD generating functional formulated in the coherent-state representation. For this derivation, we need first to give a correct expression of the generating functional in the coherent-state representation. This constitutes another purpose of this paper. Here it is necessary to mention that the corresponding path-integral expressions for the partition functions and generating functionals given in the previous literature are not correct [18–22]. The incorrectness is due to the fact that in the previous path-integral expressions, the integral representing the trace is not separated out on the one hand and the time-dependence of the integrand in the remaining part of the path integral is given incorrectly on the other hand. Such path-integral expressions can only be viewed as a formal symbolism, because in practical calculations one has to return to the original discretized forms which lead to the path-integral expressions. If one tries to perform an analytical calculation of the path integrals by employing the

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general methods and formulas of computing functional integrals, one would get a wrong result. The partition functions and generating functionals for many-body systems discussed in quantum statistics were rederived in the coherent-state representation and given correct functional-integral expressions in the author's previous paper [23]. These expressions are consistent with the corresponding coherent-state representations of the transition amplitude and the generating functional in the zero-temperature quantum theory [24–26]. Particularly, when the functional integrals are of Gaussian type, the partition functions and the generating functionals can exactly be calculated by means of the stationary-phase method [25–27]. For the case of interacting systems, the partition functions and finite-temperature Green functions can be conveniently calculated from the generating functionals by the perturbation method. The coherent-state representation of the partition function and the generating functional given in quantum statistics is now extended to thermal QCD in this paper, giving a correct formulation for the quantization of the thermal QCD in the coherent-state representation.

The remainder of this paper is arranged as follows. In Sect. 2, we quote the main results given in our previous paper for the quantum statistical mechanics. These results may straightforwardly be extended to the quantum field theory. In Sect. 3, we describe the coherent-state representation of the thermal QCD in the first order (or say, Hamiltonian) formalism. In Sect. 4, the quantization of the thermal QCD is performed in the coherent-state representation by writing out explicitly the generating functional of thermal Green functions. To demonstrate the applicability and correctness of the generating functional, we pay attention to deriving the perturbative expansion of the generating functional in the coherent-state representation. Section 5 will be used to establish the three-dimensional equation obeyed by the $q\overline{q}$ bound states at finite temperature. Section 6 serves to derive the closed expression of the interaction kernel appearing in the three-dimensional equation. In Sect. 7, the one-gluon exchange kernel and Hamiltonian will be discussed in detail. In the last section, some concluding remarks will be made. In the appendix, the perturbative expansion of the generating functional given in Sect. 4 will be transformed to the corresponding one represented in the position space.

2 Path-integral formulation of quantum statistics in the coherent-state representation

First, we start from the partition function for a grand canonical ensemble which usually is written in the form [19–22]

$$Z = \mathrm{Tr} \mathrm{e}^{-\beta \widehat{K}} , \tag{1}$$

where $\beta = \frac{1}{kT}$ with k and T being the Boltzmann constant and the temperature and

$$\widehat{K} = \widehat{H} - \mu \widehat{N} ; \tag{2}$$

here μ is the chemical potential, and $\widehat{H}$ and $\widehat{N}$ are the Hamiltonian and particle-number operators respectively. In the coherent-state representation, the trace in (1) will be represented by an integral over the coherent states. To determine the concrete form of the integral, for simplicity, let us start from an one-dimensional system. Its partition function given in the particle-number representation is

$$Z = \sum_{n=0}^{\infty} \langle n | \, \mathrm{e}^{-\beta \widehat{K}} \, |n\rangle \ . \tag{3}$$

Then we use the completeness relation of the coherent states [18–28]

$$\int D(a^* a) \mid a\rangle\langle a^* \mid = 1 , \tag{4}$$

where $\mid a\rangle$ denotes a normalized coherent state, i.e., the eigenstate of the annihilation operator $\hat{a}$ with a complex eigenvalue a [18–28]:

$$\widehat{a} \, |a\rangle = a \, |a\rangle \ , \tag{5}$$

whose Hermitian conjugate is

$$\langle a^* | \, \widehat{a}^+ = a^* \, \langle a^* | \tag{6}$$

and $D(a^* a)$ symbolizes the integration measure defined by [18–28]

$$D(a^* a) = \begin{cases} \frac{1}{\pi} \mathrm{d}a^* \mathrm{d}a, & \text{for bosons;} \\ \mathrm{d}a^* \mathrm{d}a, & \text{for fermions.} \end{cases} \tag{7}$$

In the above, we have used the eigenvalues a and a^* to designate the eigenstates $|a\rangle$ and $\langle a^*|$, respectively. It is emphasized that since we use the normalized eigenfunction of the coherent state whose expression in its own representation will be shown in (15), the completeness relation in (4) has the ordinary form as we are familiar with in quantum mechanics. Inserting (4) into (3), we have

$$Z = \sum_{n=0}^{\infty} \int D(a^* a) D(a'^* a') \, \langle n \mid a' \rangle \, \langle a'^* | \, \mathrm{e}^{-\beta \widehat{K}} \, |a\rangle \, \langle a^* \mid n \rangle \ , \tag{8}$$

where

$$\begin{aligned} \langle a^* \mid n \rangle &= \frac{1}{\sqrt{n!}} (a^*)^n \mathrm{e}^{-a^* a} , \\ \langle n \mid a' \rangle &= \frac{1}{\sqrt{n!}} (a')^n \mathrm{e}^{-a'^* a'} \end{aligned} \tag{9}$$

are the energy eigenfunctions given in the coherent-state representation (note that for fermions $n = 0, 1$) [20–26]. Both eigenfunctions commute with the matrix element $\left\langle a'^* \mid \mathrm{e}^{-\beta \widehat{K}} \mid a \right\rangle$, because the operator $\widehat{K}(\widehat{a}^+, \widehat{a})$ generally is a polynomial of the operator $\widehat{a}^+ \widehat{a}$ for fermion systems. In view of the expressions in (9) and the commutation relation [20–26]

$$a^* a' = \pm a' a^* , \tag{10}$$

where the signs "+" and "−" are attributed to bosons and fermions respectively, it is easy to see

$$\langle n \mid a' \rangle \langle a^* \mid n \rangle = \langle \pm a^* \mid n \rangle \langle n \mid a' \rangle \ . \tag{11}$$

Substituting (11) in (8) and applying the completeness relations for the particle-number states and coherent ones, one may find

$$Z = \int D(a^*a) \langle \pm a^* | \,\mathrm{e}^{-\beta\widehat{K}}\, | a \rangle \ , \tag{12}$$

where the plus and minus signs in front of a^* belong to bosons and fermions respectively.

To evaluate the matrix element in (12), we may, as usual, divide the "time"interval $[0,\beta]$ into n equal and infinitesimal parts, $\beta = n\varepsilon$. and then insert a completeness relation shown in (4) at each dividing point. In this way, (12) may be represented as [19–27]

$$\begin{aligned} Z = \int & D(a^*a) \\ & \times \prod_{i=1}^{n-1} D(a_i^* a_i) \langle \pm a^* | \,\mathrm{e}^{-\varepsilon\widehat{K}}\, | a_{n-1} \rangle \langle a_{n-1}^* | \,\mathrm{e}^{-\varepsilon\widehat{K}}\, | a_{n-2} \rangle \\ & \times \langle a_{i+1}^* | \,\mathrm{e}^{-\varepsilon\widehat{K}}\, | a_i \rangle \langle a_i^* | \,\mathrm{e}^{-\varepsilon\widehat{K}}\, | a_{i-1} \rangle \langle a_1^* | \,\mathrm{e}^{-\varepsilon\widehat{K}}\, | a \rangle \ . \end{aligned} \tag{13}$$

Since ε is infinitesimal, we can write

$$\mathrm{e}^{-\varepsilon\widehat{K}(\widehat{a}^+,\widehat{a})} \approx 1 - \varepsilon\widehat{K}(\widehat{a}^+,\widehat{a}) \,, \tag{14}$$

where $\widehat{K}(\widehat{a}^+,\widehat{a})$ is assumed to be normal-ordered. Noticing this fact, when applying (5) and (6) and the inner product of two coherent states [19–27]

$$\langle a_i^* \mid a_{i-1} \rangle = \mathrm{e}^{-\frac{1}{2}a_i^*a_i - \frac{1}{2}a_{i-1}^*a_{i-1} + a_i^*a_{i-1}} \,, \tag{15}$$

which is suitable to both bosons and fermions, one can get from (13) that

$$\begin{aligned} Z = \int & D(a^*a)\mathrm{e}^{-a^*a} \\ & \times \int \prod_{i=1}^{n-1} D(a_i^*a_i) \exp\Big\{ -\varepsilon \sum_{i=1}^{n} K(a_i^*, a_{i-1}) \\ & \quad + \sum_{i=1}^{n} a_i^* a_{i-1} - \sum_{i=1}^{n-1} a_i^* a_i \Big\} \,, \end{aligned} \tag{16}$$

where we have set

$$\pm a^* = a_n^*, \quad a = a_0 \,. \tag{17}$$

It is noted that the factor e^{-a^*a} in the first integrand comes from the matrix elements $\langle \pm a^* | \, a_{n-1} \rangle$ and $\langle a_1^* | \, a \rangle$, and the last sum in the above exponent is obtained by summing up the common terms $-\frac{1}{2}a_i^*a_i$ and $-\frac{1}{2}a_{i-1}^*a_{i-1}$ appearing in the exponents of the matrix element $\langle a_i^* \mid a_{i-1} \rangle$ and its adjacent ones $\langle a_{i+1}^* \mid a_i \rangle$ and $\langle a_{i-1}^* \mid a_{i-2} \rangle$. As will be seen in (21), such a summation is essential to give a correct time-dependence of the functional integrand in the partition function. The last two sums in (16) can be rewritten in the form

$$\begin{aligned} & \sum_{i=1}^{n} a_i^* a_{i-1} - \sum_{i=1}^{n-1} a_i^* a_i \\ & = \frac{1}{2} a_n^* a_{n-1} + \frac{1}{2} a_1^* a_0 \\ & \quad + \frac{\varepsilon}{2} \sum_{i=1}^{n-1} \left[\left(\frac{a_{i+1}^* - a_i^*}{\varepsilon} \right) a_i - a_i^* \left(\frac{a_i - a_{i-1}}{\varepsilon} \right) \right] . \end{aligned} \tag{18}$$

Upon substituting (18) in (16) and taking the limit $\varepsilon \to 0$, we obtain the path-integral expression of the partition functions as follows:

$$Z = \int D(a^*a)\mathrm{e}^{-a^*a} \int \mathfrak{D}(a^*a)\mathrm{e}^{I(a^*,a)} \,, \tag{19}$$

where

$$\mathfrak{D}(a^*a) = \begin{cases} \prod_\tau \frac{1}{\pi}\mathrm{d}a^*(\tau)\mathrm{d}a(\tau), & \text{for bosons;} \\ \prod_\tau \mathrm{d}a^*(\tau)\mathrm{d}a(\tau), & \text{for fermions} \end{cases} \tag{20}$$

and

$$\begin{aligned} I(a^*,a) &= \frac{1}{2}a^*(\beta)a(\beta) + \frac{1}{2}a^*(0)a(0) \\ & \quad - \int_0^\beta \mathrm{d}\tau \Big[\frac{1}{2}a^*(\tau)\dot{a}(\tau) - \frac{1}{2}\dot{a}^*(\tau)a(\tau) \\ & \qquad + K(a^*(\tau), a(\tau)) \Big] \\ &= a^*(\beta)a(\beta) - \int_0^\beta \mathrm{d}\tau \left[a^*(\tau)\dot{a}(\tau) + K(a^*(\tau), a(\tau)) \right] , \end{aligned} \tag{21}$$

where the last equality is obtained from the first one by a partial integration. In accordance with the definition given in (17), we see, the path-integral is subject to the following boundary conditions:

$$a^*(\beta) = \pm a^*, \quad a(0) = a \,, \tag{22}$$

where the signs "+" and "−" are written respectively for bosons and fermions. Here it is noted that (22) does not implies $a(\beta) = \pm a$ and $a^*(0) = a^*$. Actually, we have no such boundary conditions.

For systems with many degrees of freedom, the functional-integral representation of the partition functions may directly be written out from the results given in (19)–(22) as long as the eigenvalues a and a^* are understood as column matrices $a = (a_1, a_2, \cdots, a_k, \cdots)$ and $a^* = (a_1^*, a_2^*, \cdots, a_k^*, \cdots)$. Written explicitly, we have

$$Z = \int D(a^*a)\mathrm{e}^{-a_k^*a_k} \int \mathfrak{D}(a^*a)\mathrm{e}^{I(a^*,a)} \,, \tag{23}$$

where

$$D(a^*a) = \begin{cases} \prod_k \frac{1}{\pi} \mathrm{d}a_k^* \mathrm{d}a_k, & \text{for bosons;} \\ \prod_k \mathrm{d}a_k^* \mathrm{d}a_k, & \text{for fermions,} \end{cases} \tag{24}$$

$$\mathfrak{D}(a^*a) = \begin{cases} \prod_{k\tau} \frac{1}{\pi} \mathrm{d}a_k^*(\tau) \mathrm{d}a_k(\tau), & \text{for bosons;} \\ \prod_{k\tau} \mathrm{d}a_k^*(\tau) \mathrm{d}a_k(\tau), & \text{for fermions} \end{cases} \tag{25}$$

and

$$I(a^*,a) = a_k^*(\beta)a_k(\beta) - \int_0^\beta \mathrm{d}\tau \left[a_k^*(\tau)\dot{a}_k(\tau) + K\left(a_k^*(\tau), a_k(\tau)\right)\right]. \tag{26}$$

The boundary conditions in (22) now become

$$a_k^*(\beta) = \pm a_k^*, \quad a_k(0) = a_k\,. \tag{27}$$

In (23) and (26), the repeated indices imply the summations over k. If the k stands for a continuous index as in the case of quantum field theory, the summations will be replaced by integrations over k.

It should be pointed out that in the previous derivation of the coherent-state representation of the partition functions, the authors did not use the expressions given in (16) and (18). Instead, the matrix element in (15) was directly chosen to be the starting point and recast in the form [18–22]

$$\langle a_i^* | a_{i-1}\rangle = \exp\left\{ -\frac{\varepsilon}{2}\left[a_i^* \left(\frac{a_i - a_{i-1}}{\varepsilon}\right) - \left(\frac{a_i^* - a_{i-1}^*}{\varepsilon}\right) a_{i-1}\right]\right\}. \tag{28}$$

Substituting the above expression into (13) and taking the limit $\varepsilon \to 0$, it follows that [18–22]

$$Z = \int \mathfrak{D}(a^*a) \exp\left\{ -\int_0^\beta \mathrm{d}\tau \left[\frac{1}{2}a^*(\tau)\dot{a}(\tau) - \frac{1}{2}\dot{a}^*(\tau)a(\tau) + K(a^*(\tau), a(\tau))\right]\right\}. \tag{29}$$

Clearly, in the above derivation, the common terms appearing in the exponents of adjacent matrix elements were not combined together. As a result, the time-dependence of the integrand in (29) could not be given correctly. In comparison with the previous result shown in (29), the expression written in (19)–(21) has two functional integrals. The first integral which represents the trace in (1) is absent in (29). The second integral is defined as the same as the integral in (29); but the integrand are different from each other. In (19), there occur two additional factors in the integrand: one is e^{-a^*a} which comes from the initial and final states in (13), another is $\mathrm{e}^{\frac{1}{2}[a^*(\beta)a(\beta)+a^*(0)a(0)]}$ in which $a^*(\beta)$ and $a(0)$ are related to the boundary conditions shown in (22). These additional factors are also absent in (29). As will be seen soon, the occurrence of these factors in the functional-integral expression is essential to give correct calculated results.

To demonstrate the correctness of the expression given in (23)–(27), let us compute the partition function for the system whose Hamiltonian is of harmonic oscillator type as we meet in the cases of ideal gases and free fields. In this case,

$$K(a^*a) = \omega_k a_k^* a_k\,, \tag{30}$$

where $\omega_k = \varepsilon_k - \mu$ with ε_k being the particle energy, and therefore (26) becomes

$$I(a^*,a) = a_k^*(\beta)a_k(\beta) - \int_0^\beta \mathrm{d}\tau \left[a_k^*(\tau)\dot{a}_k(\tau) + \omega_k a_k^*(\tau)a_k(\tau)\right]. \tag{31}$$

By the stationary-phase method which is established based on the property of the Gaussian integral that the integral is equal to the extremum of the integrand which is an exponential function [25–27], we may write

$$\int \mathfrak{D}(a^*a)\mathrm{e}^{I(a^*,a)} = \mathrm{e}^{I_0(a^*,a)}\,, \tag{32}$$

where $I_0(a^*,a)$ is obtained from $I(a^*,a)$ by replacing the variables $a_k^*(\tau)$ and $a_k(\tau)$ in $I(a^*,a)$ with those values which are determined from the stationary condition $\delta I(a^*,a) = 0$. From this condition and the boundary conditions in (27) which implies $\delta a_k^*(\beta) = 0$ and $\delta a_k(0) = 0$, it is easy to derive the following equations of motion [24–26]:

$$\dot{a}_k(\tau) + \omega_k a_k(\tau) = 0, \quad \dot{a}_k^*(\tau) - \omega_k a_k^*(\tau) = 0\,. \tag{33}$$

Their solutions satisfying the boundary condition are

$$a_k(\tau) = a_k \mathrm{e}^{-\omega_k \tau}, \quad a_k^*(\tau) = \pm a_k^* \mathrm{e}^{\omega_k(\tau-\beta)}\,. \tag{34}$$

On substituting the above solutions into (31), we obtain

$$I_0(a^*,a) = \pm a_k^* a_k \mathrm{e}^{-\omega_k\beta}\,. \tag{35}$$

With the functional integral given in (32) and (35), the partition functions in (23) become

$$Z_0 = \begin{cases} \int D(a^*a)\mathrm{e}^{-a_k^*a_k(1-\mathrm{e}^{-\beta\omega_k})}, & \text{for bosons;} \\ \int D(a^*a)\mathrm{e}^{-a_k^*a_k(1+\mathrm{e}^{-\beta\omega_k})}, & \text{for fermions.} \end{cases} \tag{36}$$

For the boson case, the above integral can directly be calculated by employing the integration formula [18]:

$$\int D(a^*a)\mathrm{e}^{-a^*(\lambda a - b)} f(a) = \frac{1}{\lambda} f(\lambda^{-1}b) \tag{37}$$

The result is well-known, as shown in the following [20–22, 29]:

$$Z_0 = \prod_k \frac{1}{1-\mathrm{e}^{-\beta\omega_k}}\,. \tag{38}$$

For the fermion case, by using the property of a Grassmann algebra and the integration formulas [21–26]:

$$\int \mathrm{d}a = \int \mathrm{d}a^* = 0, \quad \int \mathrm{d}a^* a^* = \int \mathrm{d}a a = 1 \qquad (39)$$

it is easy to compute the integral in (36) and get the familiar result [21–23, 29]

$$Z_0 = \prod_k (1 + \mathrm{e}^{-\beta\omega_k}) . \qquad (40)$$

It is noted that if the stationary-phase method is applied to the functional integral in (29), one could not get the results as written in (38) and (40), showing the incorrectness of the previous functional-integral representation for the partition functions.

Now let us turn to discuss the general case where the Hamiltonian can be split into a free part and an interaction part. Correspondingly, we can write

$$K(a^*, a) = K_0(a^*, a) + H_{\mathrm{I}}(a^*, a) , \qquad (41)$$

where $K_0(a^*, a)$ is the same as given in (30) and $H_{\mathrm{I}}(a^*, a)$ is the interaction Hamiltonian. In this case, to evaluate the partition function, it is convenient to define a generating functional through introducing external sources $j_k^*(\tau)$ and $j_k(\tau)$ such that [21–23]

$$\begin{aligned} Z[j^*, j] &= \int D(a^* a) \mathrm{e}^{-a*_k a_k} \int \mathfrak{D}(a^* a) \exp\Big\{ a_k^*(\beta) a_k(\beta) \\ &\quad - \int_0^\beta \mathrm{d}\tau \big[a_k^*(\tau) \dot{a}_k(\tau) + K(a^* a) \\ &\quad - j_k^*(\tau) a_k(\tau) - a_k^*(\tau) j_k(\tau) \big] \Big\} \\ &= \mathrm{e}^{-\int_0^\beta \mathrm{d}\tau H_{\mathrm{I}}\left(\frac{\delta}{\delta j_k^*(\tau)}, \pm \frac{\delta}{\delta j_k(\tau)}\right)} Z_0[j^*, j] , \end{aligned} \qquad (42)$$

where the signs "+" and "−" in front of $\frac{\delta}{\delta j_k(\tau)}$ refer to bosons and fermions respectively, and $Z_0[j^*, j]$ is defined by

$$Z_0[j^*, j] = \int D(a^* a) \mathrm{e}^{-a_k^* a_k} \int \mathfrak{D}(a^* a) \mathrm{e}^{I(a^*, a; j^*, j)} , \qquad (43)$$

in which

$$\begin{aligned} I(a^*, a; j^*, j) &= a_k^*(\beta) a_k(\beta) \\ &\quad - \int_0^\beta \mathrm{d}\tau \big[a_k^*(\tau) \dot{a}_k(\tau) + \omega_k a_k^*(\tau) a_k(\tau) \\ &\quad - j_k^*(\tau) a_k(\tau) - a_k^*(\tau) j_k(\tau) \big] . \end{aligned} \qquad (44)$$

Obviously, the integral in (43) is of Gaussian type. Therefore, it can be calculated by means of the stationary-phase method as will be shown in detail in Sect. 4.

The exact partition functions can be obtained from the generating functional in (42) by setting the external sources to be zero:

$$Z = Z[j^*, j] \Big|_{j^* = j = 0} . \qquad (45)$$

In particular, the generating functional is much useful to compute the finite-temperature Green functions. For simplicity, we take the two-point Green function as an example to show this point. In many-body theory, the Green function usually is defined in the operator formalism by [21, 29]

$$\begin{aligned} G_{kl}(\tau_1, \tau_2) &= \frac{1}{Z} \mathrm{Tr} \left\{ \mathrm{e}^{-\beta \widehat{K}} T[\widehat{a}_k(\tau_1) \widehat{a}_l^+(\tau_2)] \right\} \\ &= \mathrm{Tr} \left\{ \mathrm{e}^{\beta(\Omega - \widehat{K})} T[\widehat{a}_k(\tau_1) \widehat{a}_l^+(\tau_2)] \right\} , \end{aligned} \qquad (46)$$

where $0 < \tau_1, \tau_2 < \beta$, $\Omega = -\frac{1}{\beta} \ln Z$ is the grand canonical potential, T denotes the "time" ordering operator, $\widehat{a}_k(\tau_1)$ and $\widehat{a}_l^+(\tau_2)$ represent the annihilation and creation operators respectively. According to the procedure described in (12)–(22). It is clear to see that when taking τ_1 and τ_2 at two dividing points and applying (5) and (6), the Green function may be expressed as a functional integral in the coherent-state representation as follows:

$$\begin{aligned} G_{kl}(\tau_1, \tau_2) &= \frac{1}{Z} \int D(a^* a) \mathrm{e}^{-a_k^* a_k} \\ &\quad \times \int \mathfrak{D}(a^* a) a_k(\tau_1) a_l^*(\tau_2) \mathrm{e}^{I(a^*, a)} . \end{aligned} \qquad (47)$$

With the aid of the generating functional defined in (42), the above Green function may be represented as

$$G_{kl}(\tau_1, \tau_2) = \pm \frac{1}{Z} \frac{\delta^2 Z[j^*, j]}{\delta j_k^*(\tau_1) \delta j_l(\tau_2)} \Big|_{j^* = j = 0} , \qquad (48)$$

where the sings "+" and "−" belong to bosons and fermions respectively.

3 The coherent-state representation of thermal QCD Hamiltonian and action

To write out explicitly a path-integral expression of thermal QCD in the coherent-state representation, we first need to formulate the QCD in the coherent-state representation, namely, to give exact expressions of the QCD Hamiltonian and action in the coherent-state representation. For this purpose, we only need to work with the classical fields by using some skilful treatments. Let us start from the effective Lagrangian density of QCD which appears in the path-integral of the zero-temperature QCD [22, 25, 26]

$$\begin{aligned} \mathcal{L} &= \bar{\psi} \left\{ \mathrm{i}\gamma^\mu \left(\partial_\mu - \mathrm{i}g T^a A_\mu^a \right) - m \right\} \psi - \frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a \\ &\quad - \frac{1}{2\alpha} \left(\partial^\mu A_\mu^a \right)^2 - \partial^\mu \bar{C}^a D_\mu^{ab} C^b , \end{aligned} \qquad (49)$$

where $T^a = \lambda^a/2$ is the color matrix, ψ and $\bar{\psi}$ represent the quark fields, A_μ^a are the vector potentials of gluon fields, C^a and $\bar{C}^a$ designate the ghost fields,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \qquad (50)$$

and

$$D_{\mu}^{ab}=\delta^{ab}\partial_{\mu}-gf^{abc}A_{\mu}^{c}\,. \tag{51}$$

For the sake of simplicity, we work in the Feynman gauge ($\alpha=1$). It is well-known that, in this gauge, the results obtained from the above Lagrangian are equivalent to those derived from the following Lagrangian, which is given by applying the Lorentz condition $\partial^{\mu}A_{\mu}^{a}=0$ to the Lagrangian in (49):

$$\begin{aligned}\mathcal{L}=&\bar{\psi}\left\{\mathrm{i}\gamma^{\mu}\left(\partial_{\mu}-\mathrm{i}gT^{a}A_{\mu}^{a}\right)-m\right\}\psi-\frac{1}{2}\partial_{\mu}A_{\nu}^{a}\partial^{\mu}A^{a\nu}\\&-gf^{abc}\partial_{\mu}A_{\nu}^{a}A^{b\mu}A^{c\nu}-\frac{1}{4}g^{2}f^{abc}f^{ade}A^{b\mu}A^{c\nu}A_{\mu}^{d}A_{\nu}^{e}\\&-\partial^{\mu}\bar{C}^{a}\partial_{\mu}C^{b}+gf^{abc}\partial^{\mu}\bar{C}^{a}C^{b}A_{\mu}^{c}\,.\end{aligned} \tag{52}$$

Here it is noted that the application of the Lorentz condition only changes the form of the free part of the gluon Lagrangian, remaining the interaction part of the Lagrangian in (49) formally unchanged. The above Lagrangian is written in the Minkowski metric where the γ-matrix is defined as $\gamma_0=\beta$ and $\boldsymbol{\gamma}=\beta\boldsymbol{\alpha}$ [26]. In the following, it is convenient to represent the Lagrangian in the Euclidean metric with the imaginary time $\tau=\mathrm{i}t$ where t is the real time.

Since the path-integral in (42) is established in the first order (or say, Hamiltonian) formalism, to perform the path-integral quantization of thermal QCD in the coherent-state representation, we need to recast the above Lagrangian in the first order form. In doing this, it is necessary to introduce canonical conjugate momentum densities which are defined by [26, 30]

$$\begin{aligned}\Pi_{\psi}&=\frac{\partial\mathcal{L}}{\partial\partial_{t}\psi}=\mathrm{i}\overline{\psi}\gamma^{0}=\mathrm{i}\psi^{+},\\\Pi_{\overline{\psi}}&=\frac{\partial\mathcal{L}}{\partial\partial_{t}\overline{\psi}}=0,\\\Pi_{\mu}^{a}&=\frac{\partial\mathcal{L}}{\partial\partial_{t}A^{a\mu}}=-\partial_{t}A_{\mu}^{a}+gf^{abc}A_{\mu}^{b}A_{0}^{c},\\\Pi^{a}&=\left(\frac{\partial\mathcal{L}}{\partial\partial_{t}C^{a}}\right)_{\mathrm{R}}=-\partial_{t}\overline{C}^{a},\\\overline{\Pi}^{a}&=\left(\frac{\partial\mathcal{L}}{\partial\partial_{t}\overline{C}^{a}}\right)_{\mathrm{L}}=-\partial_{t}C^{a}+gf^{abc}C^{b}A_{0}^{c}\,,\end{aligned} \tag{53}$$

where the subscripts R and L mark the right- and left-derivatives with respect to the real time respectively. With the above momentum densities, the Lagrangian in (52) can be represented as

$$\mathcal{L}=\Pi_{\psi}\partial_{t}\psi+\Pi^{a\mu}\partial_{t}A_{\mu}^{a}+\Pi^{a}\partial_{t}C^{a}+\partial_{t}\overline{C}^{a}\overline{\Pi}-\mathcal{H}\,, \tag{54}$$

where

$$\mathcal{H}=\mathcal{H}_{0}+\mathcal{H}_{\mathrm{I}} \tag{55}$$

is the Hamiltonian density in which

$$\begin{aligned}\mathcal{H}_{0}=&\bar{\psi}(\boldsymbol{\gamma}\cdot\nabla+m)\psi+\frac{1}{2}(\Pi_{\mu}^{a})^{2}-\frac{1}{2}A_{\mu}^{a}\nabla^{2}A_{\mu}^{a}\\&-\Pi^{a}\overline{\Pi}^{a}+\bar{C}^{a}\nabla^{2}C^{a}\end{aligned} \tag{56}$$

is the free Hamiltonian density and

$$\begin{aligned}\mathcal{H}_{\mathrm{I}}=&\mathrm{i}g\bar{\psi}T^{a}\gamma_{\mu}A_{\mu}^{a}\psi+gf^{abc}\left(\mathrm{i}\Pi_{\mu}^{a}A_{4}^{c}+\partial_{i}A_{\mu}^{a}A_{i}^{c}\right)A_{\mu}^{b}\\&-\frac{1}{4}g^{2}f^{abc}f^{ade}A_{\mu}^{b}A_{\mu}^{d}\left(A_{4}^{c}A_{4}^{e}-A_{i}^{c}A_{i}^{e}\right)\\&+gf^{abc}\left(\mathrm{i}\Pi^{a}A_{4}^{c}-\partial_{i}\bar{C}^{a}A_{i}^{c}\right)C^{b}\end{aligned} \tag{57}$$

is the interaction Hamiltonian density and the Latin letter i denotes the spatial index. The above Hamiltonian density is written in the Euclidean metric for later convenience. The matrix γ_{μ} in this metric is defined by $\gamma_4=\beta$ and $\boldsymbol{\gamma}=-\mathrm{i}\beta\boldsymbol{\alpha}$ [30]. It should be noted that the conjugate quantities Π^{a} and $\overline{\Pi}^{a}$ for the ghost fields are respectively defined by the right-derivative and the left one as shown in (53) because only in this way one can get correct results. This unusual definition originates from the peculiar property of the ghost fields which are scalar fields, but subject to the commutation rule of a Grassmann algebra.

In order to derive an expression of the thermal QCD in the coherent-state representation, one should employ the Fourier transformations for the canonical variables of the QCD which are listed below. For the quark field [26, 30],

$$\begin{aligned}\psi(\mathbf{x},\tau)=\int\frac{\mathrm{d}^{3}p}{(2\pi)^{3/2}}\Big[&u^{s}(\mathbf{p})b_{s}(\mathbf{p},\tau)\mathrm{e}^{\mathrm{i}\mathbf{px}}\\&+v^{s}(\mathbf{p})d_{s}^{*}(\mathbf{p},\tau)\mathrm{e}^{-\mathrm{i}\mathbf{px}}\Big]\end{aligned} \tag{58}$$

$$\begin{aligned}\overline{\psi}(\mathbf{x},\tau)=\int\frac{\mathrm{d}^{3}p}{(2\pi)^{3/2}}\Big[&\overline{u}^{s}(\mathbf{p})b_{s}^{*}(\mathbf{p},\tau)\mathrm{e}^{-\mathrm{i}\mathbf{px}}\\&+\overline{v}^{s}(\mathbf{p})d_{s}(\mathbf{p},\tau)\mathrm{e}^{\mathrm{i}\mathbf{px}}\Big]\,,\end{aligned} \tag{59}$$

where $u^{s}(\mathbf{p})$ and $v^{s}(\mathbf{p})$ are the spinor wave functions satisfying the normalization conditions $u^{s+}(\mathbf{p})u^{s}(\mathbf{p})=v^{s+}(\mathbf{p})v^{s}(\mathbf{p})=1$, $b_{s}(\mathbf{p},\tau)$ and $b_{s}^{*}(\mathbf{p},\tau)$ are the eigenvalues of the quark annihilation and creation operators $\widehat{b}_{s}(\mathbf{p},\tau)$ and $\widehat{b}_{s}^{+}(\mathbf{p},\tau)$ which are defined in the Heisenberg picture, $d_{s}(\mathbf{p},\tau)$ and $d_{s}^{*}(\mathbf{p},\tau)$ are the corresponding ones for antiquarks. For the gluon field [26, 30],

$$\begin{aligned}A_{\mu}^{c}(\mathbf{x},\tau)=\int&\frac{\mathrm{d}^{3}k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(\mathbf{k})}}\varepsilon_{\mu}^{\lambda}(\mathbf{k})\\&\times\left[a_{\lambda}^{c}(\mathbf{k},\tau)\mathrm{e}^{\mathrm{i}\mathbf{kx}}+a_{\lambda}^{c*}(\mathbf{k},\tau)\mathrm{e}^{-\mathrm{i}\mathbf{kx}}\right],\end{aligned} \tag{60}$$

where $\varepsilon_{\mu}^{\lambda}(\mathbf{k})$ is the polarization vector and

$$\begin{aligned}\Pi_{\mu}^{c}(\mathbf{x},\tau)=\mathrm{i}\int&\frac{\mathrm{d}^{3}k}{(2\pi)^{3/2}}\sqrt{\frac{\omega(\mathbf{k})}{2}}\varepsilon_{\mu}^{\lambda}(\mathbf{k})\\&\times\left[a_{\lambda}^{c}(\mathbf{k},\tau)\mathrm{e}^{\mathrm{i}\mathbf{kx}}-a_{\lambda}^{c*}(\mathbf{k},\tau)\mathrm{e}^{-\mathrm{i}\mathbf{kx}}\right],\end{aligned} \tag{61}$$

which follows from the definition in (53) and is consistent with the Fourier representation of free fields. In the above, $a_{\lambda}^{c}(\mathbf{k},\tau)$ and $a_{\lambda}^{c*}(\mathbf{k},\tau)$ are the eigenvalues of the gluon annihilation and creation operators $\widehat{a}_{\lambda}^{c}(\mathbf{k},\tau)$ and $\widehat{a}_{\lambda}^{c+}(\mathbf{k},\tau)$. For the ghost field, we have

$$\overline{C}^a(\mathbf{x},\tau)=\int\frac{\mathrm{d}^3q}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(\mathbf{q})}}\times\left[\overline{c}_a(\mathbf{q},\tau)\mathrm{e}^{\mathrm{i}\mathbf{qx}}+c_a^*(\mathbf{q},\tau)\mathrm{e}^{-\mathrm{i}\mathbf{qx}}\right],\quad(62)$$

$$C^a(\mathbf{x},\tau)=\int\frac{\mathrm{d}^3q}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(\mathbf{q})}}\times\left[c_a(\mathbf{q},\tau)\mathrm{e}^{\mathrm{i}\mathbf{qx}}+\overline{c}_a^*(\mathbf{q},\tau)\mathrm{e}^{-\mathrm{i}\mathbf{qx}}\right],\quad(63)$$

$$\Pi^a(\mathbf{x},\tau)=\mathrm{i}\int\frac{\mathrm{d}^3q}{(2\pi)^{3/2}}\sqrt{\frac{\omega(\mathbf{q})}{2}}\times\left[\overline{c}_a(\mathbf{q},\tau)\mathrm{e}^{\mathrm{i}\mathbf{qx}}-c_a^*(\mathbf{q},\tau)\mathrm{e}^{-\mathrm{i}\mathbf{qx}}\right],\quad(64)$$

and

$$\overline{\Pi}^a(\mathbf{x},\tau)=\mathrm{i}\int\frac{\mathrm{d}^3q}{(2\pi)^{3/2}}\sqrt{\frac{\omega(\mathbf{q})}{2}}\times\left[c_a(\mathbf{q},\tau)\mathrm{e}^{\mathrm{i}\mathbf{qx}}-\overline{c}_a^*(\mathbf{q},\tau)\mathrm{e}^{-\mathrm{i}\mathbf{qx}}\right],\quad(65)$$

where $c_a(\mathbf{q},\tau)$ and $c_a^*(\mathbf{q},\tau)$ are the eigenvalues of the ghost particle annihilation and creation operators $\widehat{c}_a(\mathbf{q},\tau)$ and $\widehat{c}_a^+(\mathbf{q},\tau)$ and $\overline{c}_a(\mathbf{q},\tau)$ and $\overline{c}_a^*(\mathbf{q},\tau)$ are the ones for antighost particles.

For simplifying the expressions of the Hamiltonian and action of the thermal QCD, it is convenient to use abbreviation notations. Define

$$b_s^\theta(\mathbf{p},\tau)=\begin{cases}b_s(\mathbf{p},\tau), & \text{if }\theta=+,\\ d_s^*(\mathbf{p},\tau), & \text{if }\theta=-,\end{cases}\quad(66)$$

$$W_s^\theta(\mathbf{p})=\begin{cases}(2\pi)^{-3/2}u^s(\mathbf{p}), & \text{if }\theta=+,\\ (2\pi)^{-3/2}v^s(\mathbf{p}), & \text{if }\theta=-\end{cases}\quad(67)$$

and furthermore, set $\alpha=(\mathbf{p},s,\theta)$ and

$$\sum_\alpha=\sum_{s\theta}\int\mathrm{d}^3p\,;\quad(68)$$

equations (58) and (59) may be represented as

$$\psi(\mathbf{x},\tau)=\sum_\alpha W_\alpha b_\alpha(\tau)\mathrm{e}^{\mathrm{i}\theta\mathbf{px}},$$
$$\overline{\psi}(\mathbf{x},\tau)=\sum_\alpha\overline{W}_\alpha b_\alpha^*(\tau)\mathrm{e}^{-\mathrm{i}\theta\mathbf{px}}.\quad(69)$$

Similarly, when we define

$$a_{\lambda\theta}^c(\mathbf{k},\tau)=\begin{cases}a_\lambda^c(\mathbf{k},\tau), & \text{if }\theta=+,\\ a_\lambda^{c*}(\mathbf{k},\tau), & \text{if }\theta=-,\end{cases}\quad(70)$$

$$A_{\mu\theta}^{c\lambda}(\mathbf{k})=(2\pi)^{-3/2}(2\omega(\mathbf{k}))^{-1/2}\epsilon_\mu^\lambda(\mathbf{k}),$$
$$\Pi_{\mu\theta}^{c\lambda}(\mathbf{k})=\mathrm{i}^\theta(2\pi)^{-3/2}[\omega(\mathbf{q})/2]^{1/2}\epsilon_\mu^\lambda(\mathbf{k})\quad(71)$$

and furthermore, set $\alpha=(\mathbf{k},c,\lambda,\theta)$ and

$$\sum_\alpha=\sum_{c\lambda\theta}\int\mathrm{d}^3k;\quad(72)$$

equations (60) and (61) can be written as

$$A_\mu^c(\mathbf{x},\tau)=\sum_\alpha A_\mu^\alpha a_\alpha(\tau)\mathrm{e}^{\mathrm{i}\theta\mathbf{kx}},$$
$$\Pi_\mu^c(\mathbf{x},\tau)=\sum_\alpha\Pi_\mu^\alpha a_\alpha(\tau)\mathrm{e}^{\mathrm{i}\theta\mathbf{kx}}.\quad(73)$$

For the ghost fields, if we define

$$c_\alpha^\theta(\mathbf{q},\tau)=\begin{cases}\overline{c}_a(\mathbf{q},\tau), & \text{if }\theta=+,\\ c_a^*(\mathbf{q},\tau), & \text{if }\theta=-,\end{cases}\quad(74)$$

$$G_\theta(\mathbf{q})=(2\pi)^{-3/2}[2\omega(\mathbf{q})]^{-1/2},$$
$$\Pi_\theta(\mathbf{q})=\mathrm{i}^\theta(2\pi)^{-3/2}[\omega(\mathbf{q})/2]^{1/2},\quad(75)$$

and furthermore set $\alpha=(\mathbf{q},a,\theta)$ and

$$\sum_\alpha=\sum_{a\theta}\int\mathrm{d}^3q\,;\quad(76)$$

then (62)–(65) will be expressed as

$$\overline{C}^a(\mathbf{x},\tau)=\sum_\alpha G_\alpha c_\alpha(\tau)\mathrm{e}^{\mathrm{i}\theta\mathbf{qx}},$$
$$C^a(\mathbf{x},\tau)=\sum_\alpha G_\alpha c_\alpha^*(\tau)\mathrm{e}^{-\mathrm{i}\theta\mathbf{qx}},$$
$$\Pi^a(\mathbf{x},\tau)=\sum_\alpha\Pi_\alpha c_\alpha(\tau)\mathrm{e}^{\mathrm{i}\theta\mathbf{qx}},$$
$$\overline{\Pi}^a(\mathbf{x},\tau)=\sum_\alpha\Pi_\alpha c_\alpha^*(\tau)\mathrm{e}^{-\mathrm{i}\theta\mathbf{qx}}.\quad(77)$$

Upon substituting (69), (73) and (77) into (56) and (57), it is not difficult to get

$$H_0(\tau)=\int\mathrm{d}^3x\mathcal{H}_0(x)=\sum_\alpha\theta_\alpha\varepsilon_\alpha b_\alpha^*(\tau)b_\alpha(\tau)+\frac{1}{2}\sum_\alpha\omega_\alpha a_\alpha^*(\tau)a_\alpha(\tau)+\sum_\alpha\omega_\alpha c_\alpha^*(\tau)c_\alpha(\tau)\quad(78)$$

and

$$H_{\mathrm{I}}(\tau)=\int\mathrm{d}^3x\mathcal{H}_{\mathrm{I}}(x)=\sum_{\alpha\beta\gamma}A(\alpha\beta\gamma)b_\alpha^*(\tau)b_\beta(\tau)a_\gamma(\tau)+\sum_{\alpha\beta\gamma}B(\alpha\beta\gamma)a_\alpha(\tau)a_\beta(\tau)a_\gamma(\tau)+\sum_{\alpha\beta\gamma\delta}C(\alpha\beta\gamma\delta)a_\alpha(\tau)a_\beta(\tau)a_\gamma(\tau)a_\delta(\tau)+\sum_{\alpha\beta\gamma}D(\alpha\beta\gamma)c_\alpha^*(\tau)c_\beta(\tau)a_\gamma(\tau),\quad(79)$$

which are the QCD Hamiltonian given in the coherent-state representation. In (78), the first, second and third terms are the free Hamiltonians for quarks, gluons and ghost particles respectively where $\theta_\alpha \equiv \theta$, $\varepsilon_\alpha = (\mathbf{p}^2 + m^2)^{1/2}$ is the quark energy, $\omega_\alpha = |\mathbf{k}|$ is the energy for a gluon or a ghost particle. In (79), the first term is the interaction Hamiltonian between quarks and gluons, the second and third terms are the interaction Hamiltonian among gluons and the fourth term represents the interaction Hamiltonian between ghost particles and gluons. The coefficient functions in (79) are defined as follows:

$$\begin{aligned} A(\alpha\beta\gamma) &= \mathrm{i}g(2\pi)^3\delta^3(\theta_\alpha\mathbf{p}_\alpha - \theta_\beta\mathbf{p}_\beta - \theta_\gamma\mathbf{k}_\gamma) \\ &\quad \times \overline{W}^{\theta_\alpha}_{s_\alpha}(\mathbf{p}_\alpha)T^a\gamma_\mu W^{\theta_\beta}_{s_\alpha}(\mathbf{p}_\beta)A^{a\lambda_\gamma}_{\mu\theta_\gamma}(\mathbf{k}_\gamma)\,, \qquad (80) \\ B(\alpha\beta\gamma) &= \mathrm{i}g(2\pi)^3\delta^3(\theta_\alpha\mathbf{k}_\alpha + \theta_\beta\mathbf{k}_\beta + \theta_\gamma\mathbf{k}_\gamma) \\ &\quad \times f^{abc}\Big[\Pi^{a\lambda_\alpha}_{\mu\theta_\alpha}(\mathbf{k}_\alpha)A^{c\lambda_\gamma}_{4\theta_\gamma}(\mathbf{k}_\gamma) \\ &\quad + \theta_\alpha k_i^\alpha A^{a\lambda_\alpha}_{\mu\theta_\alpha}(\mathbf{k}_\alpha)A^{c\lambda_\gamma}_{i\theta_\gamma}(\mathbf{k}_\gamma)\Big]A^{b\lambda_\beta}_{\mu\theta_\beta}(\mathbf{k}_\beta), \qquad (81) \\ C(\alpha\beta\gamma\delta) &= -\frac{1}{4}g^2(2\pi)^3\delta^3(\theta_\alpha\mathbf{k}_\alpha + \theta_\beta\mathbf{k}_\beta + \theta_\rho\mathbf{k}_\rho + \theta_\sigma\mathbf{k}_\sigma) \\ &\quad \times f^{abc}f^{ade}A^{b\lambda_\alpha}_{\mu\theta_\alpha}(\mathbf{k}_\alpha)A^{d\lambda_\beta}_{\mu\theta_\beta}(\mathbf{k}_\beta) \\ &\quad \times \Big[A^{c\lambda_\rho}_{4\theta_\rho}(\mathbf{k}_\rho)A^{e\lambda_\sigma}_{4\theta_\sigma}(\mathbf{k}_\sigma) - A^{c\lambda_\rho}_{i\theta_\rho}(\mathbf{k}_\rho)A^{e\lambda_\sigma}_{i\theta_\sigma}(\mathbf{k}_\sigma)\Big] \qquad (82) \end{aligned}$$

and

$$\begin{aligned} D(\alpha\beta\gamma) &= \mathrm{i}g(2\pi)^3\delta^3(\theta_\alpha\mathbf{q}_\alpha - \theta_\beta\mathbf{q}_\beta - \theta_\gamma\mathbf{k}_\gamma)f^{abc}G^a_{\theta_\alpha}(\mathbf{q}_\alpha) \\ &\quad \times\Big[\Pi^b_{\theta\beta}(\mathbf{q}_\beta)A^{c\lambda_\gamma}_{4\theta_\gamma}(\mathbf{k}_\gamma) - \theta_\alpha k_i^\alpha G^b_{\theta_\beta}(\mathbf{q}_\beta)A^{c\lambda_\gamma}_{i\theta_\gamma}(\mathbf{k}_\gamma)\Big]\,. \qquad (83) \end{aligned}$$

It is emphasized that the expressions in (78) and (79) are just the Hamiltonian of QCD appearing in the path-integral as shown in (42), where all the creation and annihilation operators in the Hamiltonian (which are written in a normal product) are replaced by their eigenvalues.

To write the path-integral of thermal QCD, we need also an expression for the action S given in the coherent-state representation. This action can be obtained by using the Lagrangian density shown in (54). By partial integration and considering the following boundary conditions of the fields [20–22]:

$$\begin{aligned} &\psi(\mathbf{x},0) = \psi(\mathbf{x}), \qquad \overline{\psi}(\mathbf{x},0) = \overline{\psi}(\mathbf{x}), \\ &\psi(\mathbf{x},\beta) = -\psi(\mathbf{x}), \qquad \overline{\psi}(\mathbf{x},\beta) = -\overline{\psi}(\mathbf{x}), \qquad (84) \\ &A^c_\mu(\mathbf{x},0) = A^c_\mu(\mathbf{x},\beta) = A^c_\mu(\mathbf{x}), \\ &\Pi^c_\mu(\mathbf{x},0) = \Pi^c_\mu(\mathbf{x},\beta) = \Pi^c_\mu(\mathbf{x}) \qquad (85) \end{aligned}$$

and

$$\begin{aligned} &\overline{C}^a(\mathbf{x},0) = \overline{C}^a(\mathbf{x},\beta) = \overline{C}^a(\mathbf{x}), \\ &C^a(\mathbf{x},0) = C^a(\mathbf{x},\beta) = C^a(\mathbf{x}), \\ &\overline{\Pi}^a(\mathbf{x},0) = \overline{\Pi}^a(\mathbf{x},\beta) = \overline{\Pi}^a(\mathbf{x}), \\ &\Pi^a(\mathbf{x},0) = \Pi^a(\mathbf{x},\beta) = \Pi^a(\mathbf{x}); \qquad (86) \end{aligned}$$

the action given by the Lagrangian density in (54) can be represented in the form

$$\begin{aligned} S = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\Big\{&\frac{1}{2}[\psi^+(\mathbf{x},\tau)\dot{\psi}(\mathbf{x},\tau) - \dot{\psi}^+(\mathbf{x},\tau)\psi(\mathbf{x},\tau)] \\ &+ \frac{\mathrm{i}}{2}\Big[\Pi^c_\mu(\mathbf{x},\tau)\dot{A}^c_\mu(\mathbf{x},\tau) - \dot{\Pi}^c_\mu(\mathbf{x},\tau)A^c_\mu(\mathbf{x},\tau)\Big] \\ &+ \frac{\mathrm{i}}{2}\Big[\Pi_a(\mathbf{x},\tau)\dot{C}_a(\mathbf{x},\tau) - \dot{\Pi}_a(\mathbf{x},\tau)C_a(\mathbf{x},\tau) \\ &+ \dot{\overline{C}}_a(\mathbf{x},\tau)\overline{\Pi}_a(\mathbf{x},\tau) - \overline{C}_a(\mathbf{x},\tau)\dot{\overline{\Pi}}_a(\mathbf{x},\tau)\Big] - \mathcal{H}(\mathbf{x},\tau)\Big\}\,, \qquad (87) \end{aligned}$$

where the first relation in (53) has been used, and the symbol "$\cdot$" in $\dot{\psi}(\mathbf{x},\tau)$, $\dot{A}^c_\mu(\mathbf{x},\tau)\cdots$ now denotes the derivatives of the fields with respect to the imaginary time τ. It is stressed here that only the above expression is appropriate to use for deriving the coherent-state representation of the action by making use of the Fourier expansions written in (58)–(65). On inserting (58)–(65) into (87), it is not difficult to get

$$\begin{aligned} S = -\int_0^\beta \mathrm{d}\tau\Big\{\int \mathrm{d}^3k\Big\{&\frac{1}{2}\Big[b^*_s(\mathbf{k},\tau)\dot{b}_s(\mathbf{k},\tau) - \dot{b}^*_s(\mathbf{k},\tau)b_s(\mathbf{k},\tau)\Big] \\ &+ \frac{1}{2}\Big[d^*_s(\mathbf{k},\tau)\dot{d}_s(\mathbf{k},\tau) - \dot{d}^*_s(\mathbf{k},\tau)d_s(\mathbf{k},\tau)\Big] \\ &+ \frac{1}{2}[a^{c*}_\lambda(\mathbf{k},\tau)\dot{a}^c_\lambda(\mathbf{k},\tau) - \dot{a}^{c*}_\lambda(\mathbf{k},\tau)a^c_\lambda(\mathbf{k},\tau)] \\ &+ \frac{1}{2}\Big[\overline{c}^*_a(\mathbf{k},\tau)\dot{\overline{c}}_a(\mathbf{k},\tau) - \dot{\overline{c}}^*_a(\mathbf{k},\tau)\overline{c}_a(\mathbf{k},\tau) \\ &- c^*_a(\mathbf{k},\tau)\dot{c}_a(\mathbf{k},\tau) + \dot{c}^*_a(\mathbf{k},\tau)c_a(\mathbf{k},\tau)]\Big\} + H(\tau)\Big\} \\ = -S_\mathrm{E}\,, \qquad (88) \end{aligned}$$

where $H(\tau)$ is given by the sum of the Hamiltonians in (78) and (79) and S_E is the action defined in the Euclidean metric. It is noted that if one considers a grand canonical ensemble of QCD, the Hamiltonian in (88) should be replaced by $K(\tau)$ defined in (2). Employing the abbreviation notation as denoted in (66), (70) and (74) and letting q_α stand for $(a_\alpha, b_\alpha, c_\alpha)$, the action may be compactly represented as

$$S_\mathrm{E} = \int_0^\beta \mathrm{d}\tau\left\{\sum_\alpha \frac{1}{2}[q^*_\alpha(\tau)\circ\dot{q}_\alpha(\tau) - \dot{q}^*_\alpha(\tau)\circ q_\alpha(\tau)] + H(\tau)\right\}, \qquad (89)$$

where we have defined

$$q^*_\alpha \circ q_\alpha = a_{\alpha^-}a_{\alpha^+} + b^*_\alpha b_\alpha + \theta_\alpha c^*_\alpha c_\alpha\,. \qquad (90)$$

It is emphasized that the $\theta_\alpha = \pm$ is now contained in the subscript α. Therefore, each α may take the value α^+ and/or α^- as the first term in (90) does.

4 Generating functional of Green functions for thermal QCD

With the action S_{E} given in the preceding section, the quantization of the thermal QCD in the coherent-state representation is easily implemented by writing out its generating functional of thermal Green functions. According to the general formula shown in (42), the QCD generating functional can be formulated as

$$Z[j]=\int D(q^*\cdot q)\mathrm{e}^{-q^*\cdot q}\int \mathfrak{D}(q^*\cdot q)\exp\left\{\frac{1}{2}[q^*(\beta)\cdot q(\beta) - q^*(0)\cdot q(0)]-S_{\mathrm{E}}+\int_0^\beta \mathrm{d}\tau j^*(\tau)\cdot q(\tau)\right\}, \quad (91)$$

where we have defined

$$q^*\cdot q=\frac{1}{2}a_\alpha^* a_\alpha+\theta_\alpha b_\alpha^* b_\alpha+c_\alpha^* c_\alpha \quad (92)$$

and

$$j^*\cdot q=\xi_\alpha^* a_\alpha+\theta_\alpha(\eta_\alpha^* b_\alpha+b_\alpha^*\eta_\alpha+\zeta_\alpha^* c_\alpha+c_\alpha^*\zeta_\alpha) \quad (93)$$

here ξ_α,η_α and ζ_α are the sources for gluons, quarks and ghost particles respectively and the repeated index implies summation. It is noted that the product $q^*\cdot q$ defined above is different from the $q_\alpha^*\circ q_\alpha$ defined in (90) in the terms for quarks and ghost particles and the subscript α in (92) and (93) is also defined by containing $\theta_\alpha=\pm$. In what follows, we assign $\alpha^\pm$ to represent the α with $\theta_\alpha=\pm$. According to this notation, the sources in (93) are specifically defined as follows:

$$\begin{aligned}&\xi_{\alpha^+}=\xi_\alpha,\ \xi_{\alpha^-}=\xi_\alpha^*\\ &\eta_{\alpha^+}=\eta_\alpha,\ \eta_{\alpha^-}=\overline{\eta}_\alpha^*\\ &\zeta_{\alpha^+}=\zeta_\alpha,\ \zeta_{\alpha^-}=\overline{\zeta}_\alpha^*,\end{aligned} \quad (94)$$

where the subscript α on the right hand side of each equality no longer contains θ_α and the gluon term in (92) $(1/2)a_\alpha^* a_\alpha$ may be replaced by $a_{\alpha^-}a_{\alpha^+}$. The integration measures $D(q^*q)$ and $\mathfrak{D}(q^*q)$ are defined as in (24) and (25).

The generating functional in (91) is nonperturbative. Now we are interested in describing the perturbation method of calculating the QCD generating functional. Since the Hamiltonian can be split into two parts $H_0(\tau)$ and $H_{\mathrm{I}}(\tau)$ as shown in (78) and (79), the generating functional in (91) may be perturbatively represented in the form

$$Z[j]=\exp\left\{-\int_0^\beta \mathrm{d}\tau H_{\mathrm{I}}\left(\frac{\delta}{\delta j(\tau)}\right)\right\}Z^0[j], \quad (95)$$

where $Z^0[j]$ is the generating functional for the free system and the exponential may be expanded in a Taylor series. In the above, the commutativity between H_{I} and $Z^0[j]$ has been considered. Obviously, the $Z^0[j]$ can be written as

$$Z^0[j]=Z_g^0[\xi]Z_q^0[\eta]Z_c^0[\zeta], \quad (96)$$

where $Z_g^0[\xi]$, $Z_q^0[\eta]$ and $Z_c^0[\zeta]$ are the generating functionals contributed from the free Hamiltonians of gluons, quarks and ghost particles respectively. They are separately and specifically described below.

In view of the expressions in (91), (88) and (78), the generating functional $Z_g^0[\xi]$ is of the form

$$Z_g^0[\xi]=\int D(a^*a)\exp\left\{-\int \mathrm{d}^3k a_\lambda^*(\mathbf{k})a_\lambda(\mathbf{k})\right\}\times\int\mathfrak{D}(a^*a)\exp\left\{I_g\left(a_\lambda^*,a_\lambda;\xi_\lambda^*,\xi_\lambda\right)\right\}, \quad (97)$$

where

$$\begin{aligned}&I_g\left(a_\lambda^*,a_\lambda;\xi_\lambda^*,\xi_\lambda\right)\\ &=\int \mathrm{d}^3k\frac{1}{2}\left[a_\lambda^*(\mathbf{k},\beta)a_\lambda(\mathbf{k},\beta)+a_\lambda^*(\mathbf{k},0)a_\lambda(\mathbf{k},0)\right]\\ &-\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3k\Big\{\frac{1}{2}\left[a_\lambda^*(\mathbf{k},\tau)\dot a_\lambda(\mathbf{k},\tau)-\dot a_\lambda^*(\mathbf{k},\tau)a_\lambda(\mathbf{k},\tau)\right]\\ &+\omega(\mathbf{k})a_\lambda^*(\mathbf{k},\tau)a_\lambda(\mathbf{k},\tau)\\ &-\xi_\lambda^*(\mathbf{k},\tau)a_\lambda(\mathbf{k},\tau)-a_\lambda^*(\mathbf{k},\tau)\xi_\lambda(\mathbf{k},\tau)\Big\}\end{aligned} \quad (98)$$

and

$$\begin{aligned}&D(a^*a)=\prod_{\mathbf{k}\lambda}\frac{1}{\pi}\mathrm{d}a_\lambda^*(\mathbf{k})\mathrm{d}a_\lambda(\mathbf{k}),\\ &\mathfrak{D}(a^*a)=\prod_{\mathbf{k}\lambda\tau}\frac{1}{\pi}\mathrm{d}a_\lambda^*(\mathbf{k},\tau)\mathrm{d}a_\lambda(\mathbf{k},\tau).\end{aligned} \quad (99)$$

The subscript λ in the above is now assigned to denote polarization and color. When we perform a partial integration, (98) becomes

$$\begin{aligned}&I_g(a_\lambda^*,a_\lambda;\xi_\lambda^*,\xi_\lambda)\\ &=\int \mathrm{d}^3k a_\lambda^*(\mathbf{k},\beta)a_\lambda(\mathbf{k},\beta)-\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3k\Big\{a_\lambda^*(\mathbf{k},\tau)\dot a_\lambda(\mathbf{k},\tau)\\ &+\omega(\mathbf{k})a_\lambda^*(\mathbf{k},\tau)a_\lambda(\mathbf{k},\tau)-\xi_\lambda^*(\mathbf{k},\tau)a_\lambda(\mathbf{k},\tau)\\ &-a_\lambda^*(\mathbf{k},\tau)\xi_\lambda(\mathbf{k},\tau)\Big\}.\end{aligned} \quad (100)$$

For the generating functional $Z_q^0[\eta]$, we can write

$$\begin{aligned}Z_q^0[\eta]=&\int D(b^*bd^*d)\\ &\times\exp\left\{-\int \mathrm{d}^3k\left[b_s^*(\mathbf{k})b_s(\mathbf{k})+d_s^*(\mathbf{k})d_s(\mathbf{k})\right]\right\}\\ &\times\int\mathfrak{D}(b^*bd^*d)\exp\left\{I_q\left(b_s^*,b_s,d_s^*,d_s;\eta_s^*,\eta_s,\overline{\eta}_s^*,\overline{\eta}_s\right)\right\},\end{aligned} \quad (101)$$

where

$$\begin{aligned}&I_q\left(b_s^*,b_s,d_s^*,d_s;\eta_s^*,\eta_s,\overline{\eta}_s^*,\overline{\eta}_s\right)\\ &=\int \mathrm{d}^3k\frac{1}{2}\big[b_s^*(\mathbf{k},\beta)b_s(\mathbf{k},\beta)+d_s^*(\mathbf{k},\beta)d_s(\mathbf{k},\beta)\\ &+b_s^*(\mathbf{k},0)b_s(\mathbf{k},0)+d_s^*(\mathbf{k},0)d_s(\mathbf{k},0)\big]\end{aligned}$$

$$-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3k\Big\{\frac{1}{2}\big[b_s^*(\mathbf{k},\tau)\dot{b}_s(\mathbf{k},\tau)-\dot{b}_s^*(\mathbf{k},\tau)b_s(\mathbf{k},\tau)\big] + \frac{1}{2}\big[d_s^*(\mathbf{k},\tau)\dot{d}_s(\mathbf{k},\tau)-\dot{d}_s^*(\mathbf{k},\tau)d_s(\mathbf{k},\tau)\big] + \varepsilon(\mathbf{k})\big[b_s^*(\mathbf{k},\tau)b_s(\mathbf{k},\tau)+d_s^*(\mathbf{k},\tau)d_s(\mathbf{k},\tau)\big] - \big[\eta_s^*(\mathbf{k},\tau)b_s(\mathbf{k},\tau)+b_s^*(\mathbf{k},\tau)\eta_s(\mathbf{k},\tau)+\overline{\eta}_s^*(\mathbf{k},\tau)d_s(\mathbf{k},\tau) + d_s^*(\mathbf{k},\tau)\overline{\eta}_s\mathbf{k},\tau]\Big\} \quad (102)$$

and

$$D(b^*bd^*d)=\prod_{\mathbf{k}s}\mathrm{d}b_s^*(\mathbf{k})\mathrm{d}b_s(\mathbf{k})\mathrm{d}d_s^*(\mathbf{k})\mathrm{d}d_s(\mathbf{k}),$$
$$\mathfrak{D}(b^*bd^*d)=\prod_{\mathbf{k}s\tau}\mathrm{d}b_s^*(\mathbf{k},\tau)\mathrm{d}b_s(\mathbf{k},\tau)\mathrm{d}d_s^*(\mathbf{k},\tau)\mathrm{d}d_s(\mathbf{k},\tau)\,, \quad (103)$$

in which the subscript s stands for spin, color and flavor. By a partial integration over τ, (102) may be given a simpler expression:

$$I_q\left(b_s^*,b_s,d_s^*,d_s;\eta_s^*,\eta_s,\overline{\eta}_s^*,\overline{\eta}_s\right) = \int \mathrm{d}^3k\big[b_s^*(\mathbf{k},\beta)b_s(\mathbf{k},\beta)+d_s^*(\mathbf{k},\beta)d_s(\mathbf{k},\beta)\big] - \int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3k\Big\{b_s^*(\mathbf{k},\tau)\dot{b}_s(\mathbf{k},\tau)+d_s^*(\mathbf{k},\tau)\dot{d}_s(\mathbf{k},\tau) + \varepsilon(\mathbf{k})\big[b_s^*(\mathbf{k},\tau)b_s(\mathbf{k},\tau)+d_s^*(\mathbf{k},\tau)d_s(\mathbf{k},\tau)\big] - \big[\eta_s^*(\mathbf{k},\tau)b_s(\mathbf{k},\tau)+b_s^*(\mathbf{k},\tau)\eta_s(\mathbf{k},\tau)+\overline{\eta}_s^*(\mathbf{k},\tau)d_s(\mathbf{k},\tau) + d_s^*(\mathbf{k},\tau)\overline{\eta}_s\mathbf{k},\tau]\Big\}. \quad (104)$$

As for the generating functional $Z_c^0[\zeta]$, we have

$$Z_c^0[\zeta]=\int D(\overline{c}^*\overline{c}cc^*) \times\exp\left\{-\int \mathrm{d}^3k\big[\overline{c}_a^*(\mathbf{k})\overline{c}_a(\mathbf{k})-c_a^*(\mathbf{k})c_a(\mathbf{k})\big]\right\} \times\int\mathfrak{D}(\overline{c}^*\overline{c}cc^*)\exp\Big\{I_c\big(c_a^*,c_a,\overline{c}_a^*,\overline{c}_a;\zeta_a^*,\zeta_a,\overline{\zeta}_a^*,\overline{\zeta}_a\big)\Big\}\,, \quad (105)$$

where

$$I_c\big(c_a^*,c_a,\overline{c}_a^*,\overline{c}_a;\zeta_a^*,\zeta_a,\overline{\zeta}_a^*,\overline{\zeta}_a\big) = \int \mathrm{d}^3k\frac{1}{2}\big[\overline{c}_a^*(\mathbf{k},\beta)\overline{c}_a(\mathbf{k},\beta)-c_a^*(\mathbf{k},\beta)c_a(\mathbf{k},\beta) + \overline{c}_a^*(\mathbf{k},0)\overline{c}_a(\mathbf{k},0)-c_a^*(\mathbf{k},0)c_a(\mathbf{k},0)\big] - \int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3k\Big\{\frac{1}{2}\big[\overline{c}_a^*(\mathbf{k},\tau)\dot{\overline{c}}_a(\mathbf{k},\tau)-\dot{\overline{c}}_a^*(\mathbf{k},\tau)\overline{c}_a(\mathbf{k},\tau)\big] - \frac{1}{2}\big[c_a^*(\mathbf{k},\tau)\dot{c}_a(\mathbf{k},\tau)-\dot{c}_a^*(\mathbf{k},\tau)c_a(\mathbf{k},\tau)\big] + \omega(\mathbf{k})\big[\overline{c}_a^*(\mathbf{k},\tau)\overline{c}_a(\mathbf{k},\tau)-c_a^*(\mathbf{k},\tau)c_a(\mathbf{k},\tau)\big] - \big[\zeta_a^*(\mathbf{k},\tau)c_a(\mathbf{k},\tau)+c_a^*(\mathbf{k},\tau)\zeta_a(\mathbf{k},\tau) + \overline{\zeta}_a^*(\mathbf{k},\tau)\overline{c}_a(\mathbf{k},\tau)+\overline{c}_a^*(\mathbf{k},\tau)\overline{\zeta}_a(\mathbf{k},\tau)\big]\Big\} \quad (106)$$

and

$$D(\overline{c}^*\overline{c}cc^*)=\prod_{\mathbf{k}a}d\overline{c}_a^*(\mathbf{k})d\overline{c}_a(\mathbf{k})\mathrm{d}c_a(\mathbf{k})\mathrm{d}c_a^*(\mathbf{k}),$$
$$\mathfrak{D}(\overline{c}^*\overline{c}cc^*)=\prod_{\mathbf{k}a\tau}\mathrm{d}\overline{c}_a^*(\mathbf{k},\tau)\mathrm{d}\overline{c}_a(\mathbf{k},\tau)\mathrm{d}c_a(\mathbf{k},\tau)\mathrm{d}c_a^*(\mathbf{k},\tau)\,, \quad (107)$$

in which the subscript a is a color index. After a partial integration, (106) is reduced to

$$I_c\big(c_a^*,c_a,\overline{c}_a^*,\overline{c}_a;\zeta_a^*,\zeta_a,\overline{\zeta}_a^*,\overline{\zeta}_a\big) = \int\mathrm{d}^3k\big[\overline{c}_a^*(\mathbf{k},\beta)\overline{c}_a(\mathbf{k},\beta)-c_a^*(\mathbf{k},\beta)c_a(\mathbf{k},\beta)\big] - \int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3k\Big\{\overline{c}_a^*(\mathbf{k},\tau)\dot{\overline{c}}_a(\mathbf{k},\tau)-c_a^*(\mathbf{k},\tau)\dot{c}_a(\mathbf{k},\tau) + \omega(\mathbf{k})\big[\overline{c}_a^*(\mathbf{k},\tau)\overline{c}_a(\mathbf{k},\tau)-c_a^*(\mathbf{k},\tau)c_a(\mathbf{k},\tau)\big] - \big[\zeta_a^*(\mathbf{k},\tau)c_a(\mathbf{k},\tau)+c_a^*(\mathbf{k},\tau)\zeta_a(\mathbf{k},\tau) + \overline{\zeta}_a^*(\mathbf{k},\tau)\overline{c}_a(\mathbf{k},\tau)+\overline{c}_a^*(\mathbf{k},\tau)\overline{\zeta}_a(\mathbf{k},\tau)\big]\Big\}. \quad (108)$$

Here it is noted that all the terms related to the quantities c_a^* and c_a are opposite in sign to the terms related to the $\overline{c}_a^*$ and $\overline{c}_a$ and, correspondingly, the definitions of the integration measures for these quantities, as shown in (107), are different from each other in the order of the differentials.

The generating functionals in (97), (101) and (105) are all of Gaussian type; therefore, they can exactly be calculated by the stationary-phase method. First, we calculate the functional integral $Z_g^0[\xi]$. According to the stationary-phase method, the functional $Z_g^0[\xi]$ can be represented in the form

$$Z_g^0[\xi]=\int D(a^*a) \times\exp\left\{-\int\mathrm{d}^3k\big[a_\lambda^*(\mathbf{k})a_\lambda(\mathbf{k})+I_g^0(a_\lambda^*,a_\lambda;\xi_\lambda^*,\xi_\lambda)\big]\right\}\,, \quad (109)$$

where $I_g^0(a_\lambda^*,a_\lambda;\xi_\lambda^*,\xi_\lambda)$ is given by the stationary condition $\delta I_g(a_\lambda^*,a_\lambda;\xi_\lambda^*,\xi_\lambda)=0$. By this condition and the boundary condition [20–22],

$$a_\lambda^*(\mathbf{k},\beta)=a_\lambda^*(\mathbf{k}),\quad a_\lambda(\mathbf{k},0)=a_\lambda(\mathbf{k}), \quad (110)$$

one may derive from (98) or (100) the following inhomogeneous equations of motion [23–26]:

$$\dot{a}_\lambda(\mathbf{k},\tau)+\omega(\mathbf{k})a_\lambda(\mathbf{k},\tau)=\xi_\lambda(\mathbf{k},\tau),$$
$$\dot{a}_\lambda^*(\mathbf{k},\tau)-\omega(\mathbf{k})a_\lambda^*(\mathbf{k},\tau)=-\xi_\lambda^*(\mathbf{k},\tau). \quad (111)$$

In accordance with the general method of solving such a kind of equations, one may first solve the homogeneous linear equations as written in (33). Based on the solutions shown in (34) and the boundary condition denoted in (110), one may assume [23–26]

$$a_\lambda(\mathbf{k},\tau)=[a_\lambda(\mathbf{k})+u_\lambda(\mathbf{k},\tau)]\mathrm{e}^{-\omega(\mathbf{k})\tau},$$
$$a_\lambda^*(\mathbf{k},\tau)=\big[a_\lambda^*(\mathbf{k})+u_\lambda^*(\mathbf{k},\tau)\big]\mathrm{e}^{\omega(\mathbf{k})(\tau-\beta)}\,, \quad (112)$$

where the unknown functions $u_\lambda(\mathbf{k},\tau)$ and $u_\lambda^*(\mathbf{k},\tau)$ are required to satisfy the boundary conditions [23–26]:

$$u_\lambda(\mathbf{k},0)=u_\lambda(\mathbf{k},\beta)=u_\lambda^*(\mathbf{k},0)=u_\lambda^*(\mathbf{k},\beta)=0. \tag{113}$$

Inserting (112) into (111), we find

$$\begin{aligned}\dot u_\lambda(\tau)&=\xi_\lambda(\mathbf{k},\tau)\mathrm{e}^{\omega(\mathbf{k})\tau}.\\ \dot u_\lambda^*(\tau)&=-\xi_\lambda^*(\mathbf{k},\tau)\mathrm{e}^{\omega(\mathbf{k})(\beta-\tau)}.\end{aligned} \tag{114}$$

Integrating these two equations and applying the boundary conditions in (113), one can get

$$\begin{aligned}u_\lambda(\mathbf{k},\tau)&=\int_0^\tau \mathrm{d}\tau'\mathrm{e}^{\omega(\mathbf{k})\tau'}\xi_\lambda(\mathbf{k},\tau'),\\ u_\lambda^*(\mathbf{k},\tau)&=-\int_\beta^\tau \mathrm{d}\tau'\mathrm{e}^{\omega(\mathbf{k})(\beta-\tau')}\xi_\lambda^*(\mathbf{k},\tau').\end{aligned} \tag{115}$$

Substitution of these solutions in (112) yields [23–26]

$$\begin{aligned}a_\lambda(\mathbf{k},\tau)&=a_\lambda(\mathbf{k})\mathrm{e}^{-\omega(\mathbf{k})\tau}+\int_0^\tau \mathrm{d}\tau'\mathrm{e}^{-\omega(\mathbf{k})(\tau-\tau')}\xi_\lambda(\mathbf{k},\tau'),\\ a_\lambda^*(\mathbf{k},\tau)&=a_\lambda^*(\mathbf{k})\mathrm{e}^{\omega(\mathbf{k})(\tau-\beta)}+\int_\tau^\beta \mathrm{d}\tau'\mathrm{e}^{\omega(\mathbf{k})(\tau-\tau')}\xi_\lambda^*(\mathbf{k},\tau').\end{aligned} \tag{116}$$

When (116) is inserted into (98) or (100), one may obtain the $I_g^0(a_\lambda^*,a_\lambda;\xi_\lambda^*,\xi_\lambda)$ which leads to another expression of (109) like this

$$\begin{aligned}&Z_g^0[\xi]\\ &=\int D(a^*a)\exp\Bigg\{-\int \mathrm{d}^3k\Big[a_\lambda^*(\mathbf{k})a_\lambda(\mathbf{k})(1-\mathrm{e}^{-\beta\omega(\mathbf{k})})\\ &\quad -a_\lambda^*(\mathbf{k})\mathrm{e}^{-\beta\omega(\mathbf{k})}\int_0^\beta \mathrm{d}\tau\mathrm{e}^{\omega(\mathbf{k})\tau}\xi_\lambda(\mathbf{k},\tau)\\ &\quad -\int_0^\beta \mathrm{d}\tau\mathrm{e}^{-\omega(\mathbf{k})\tau}\xi_\lambda^*(\mathbf{k},\tau)a_\lambda(\mathbf{k})\Big]+\int_0^\beta \mathrm{d}\tau_1\int_0^\beta \mathrm{d}\tau_2\\ &\quad\times\int \mathrm{d}^3k\xi_\lambda^*(\mathbf{k},\tau_1)\theta(\tau_1-\tau_2)\mathrm{e}^{-\omega(\mathbf{k})(\tau_1-\tau_2)}\xi_\lambda(\mathbf{k},\tau_2)\Bigg\}.\end{aligned} \tag{117}$$

When we set

$$\begin{aligned}\lambda&=1-\mathrm{e}^{-\beta\omega(\mathbf{k})},\\ b&=\mathrm{e}^{-\beta\omega(\mathbf{k})}\int_0^\beta \mathrm{d}\tau\mathrm{e}^{\omega(\mathbf{k})\tau}\xi_\lambda(\mathbf{k},\tau),\\ f(a)&=\int_0^\beta \mathrm{d}\tau\mathrm{e}^{-\omega(\mathbf{k})\tau}\xi_\lambda^*(\mathbf{k},\tau)a_\lambda(\mathbf{k}),\end{aligned} \tag{118}$$

by employing the formula denoted in (37), the integral over $a_\lambda^*(\mathbf{k})$ and $a_\lambda(\mathbf{k})$ in (117) can easily be calculated. The result is

$$\begin{aligned}Z_g^0[\xi]=Z_g^0\exp\Bigg\{&-\int_0^\beta \mathrm{d}\tau_1\int_0^\beta \mathrm{d}\tau_2\\ &\times\int \mathrm{d}^3k\xi_a^{\lambda*}(\mathbf{k},\tau_1)\Delta_{\lambda\lambda'}^{aa'}(\mathbf{k},\tau_1-\tau_2)\xi_{a'}^{\lambda'}(\mathbf{k},\tau_2)\Bigg\},\end{aligned} \tag{119}$$

where

$$Z_g^0=\prod_{\mathbf{k}\lambda a}\left[1-\mathrm{e}^{-\beta\omega(\mathbf{k})}\right]^{-1}=\prod_{\mathbf{k}a}\left[1-\mathrm{e}^{-\beta\omega(\mathbf{k})}\right]^{-4} \tag{120}$$

is precisely the partition function contributed from the free gluons [21–24, 29] and

$$\Delta_{\lambda\lambda'}^{aa'}(\mathbf{k},\tau_1-\tau_2)=g_{\lambda\lambda'}\delta^{aa'}\Delta_g(\mathbf{k},\tau_1-\tau_2), \tag{121}$$

where

$$\Delta_g(\mathbf{k},\tau_1-\tau_2)=\theta(\tau_1-\tau_2)-\left(1-\mathrm{e}^{\beta\omega(\mathbf{k})}\right)^{-1}\mathrm{e}^{-\omega(\mathbf{k})(\tau_1-\tau_2)} \tag{122}$$

is the free gluon propagator given in the Feynman gauge and in the Minkowski metric (note that in Euclidean metric, $g_{\lambda\lambda'}\to-\delta_{\lambda\lambda'}$). In (119), the color index a has been explicitly written out and the λ now merely designates the polarization index. In the other expressions, we still use λ to mark the indices of both color and polarization for simplicity. When we interchange the integration variables τ_1 and τ_2 and make a transformation $\mathbf{k}\to-\mathbf{k}$ in (119), by considering the relation

$$\xi_\lambda^*(\mathbf{k},\tau)=\xi_\lambda(-\mathbf{k},\tau), \tag{123}$$

which will be interpreted in the appendix, one may find that the propagator in (122) can be represented in the form

$$\Delta_g(\mathbf{k},\tau_1-\tau_2)=\frac{1}{2}\left[\overline{n}_b(\mathbf{k})\mathrm{e}^{-\omega(\mathbf{k})|\tau_1-\tau_2|}-n_b(\mathbf{k})\mathrm{e}^{\omega(\mathbf{k})|\tau_1-\tau_2|}\right], \tag{124}$$

where

$$\overline{n}_b(\mathbf{k})=\left(1-\mathrm{e}^{-\beta\varepsilon(\mathbf{k})}\right)^{-1},\quad n_b(\mathbf{k})=\left(1-\mathrm{e}^{\beta\varepsilon(\mathbf{k})}\right)^{-1} \tag{125}$$

are just the boson distribution functions [20–23, 29].

Let us turn to the calculation of the functional integral in (101). Based on the stationary-phase method, we can write

$$\begin{aligned}Z_q^0[\eta]=\int D(b^*bd^*d)&\\ &\times\exp\left\{-\int \mathrm{d}^3k\left[b_s^*(\mathbf{k})b_s(\mathbf{k})+d_s^*(\mathbf{k})d_s(\mathbf{k})\right]\right\}\\ &\times\exp\left\{I_q^0\left(b_s^*,b_s,d_s^*,d_s;\eta_s^*,\eta_s,\overline{\eta}_s^*,\overline{\eta}_s\right)\right\},\end{aligned} \tag{126}$$

where $I_q^0(b_s^*,b_s,d_s^*,d_s;\eta_s^*,\eta_s,\overline{\eta}_s^*,\overline{\eta}_s)$ will be obtained from (102) or (104) by the stationary condition $\delta I_q(b_s^*,b_s,d_s^*,d_s;$

$\eta^*_s, \eta_s, \overline{\eta}^*_s, \overline{\eta}_s) = 0$. From this condition and the boundary conditions [21–26]

$$
\begin{aligned}
&b^*_s(\mathbf{k}, \beta) = -b^*_s(\mathbf{k}),\ b_s(\mathbf{k}, 0) = b_s(\mathbf{k})\,,\\
&d^*_s(\mathbf{k}, \beta) = -d^*_s(\mathbf{k}),\ d_s(\mathbf{k}, 0) = d_s(\mathbf{k})\,,
\end{aligned} \tag{127}
$$

one may deduce from (102) or (104) the following equations [23–26]:

$$
\begin{aligned}
&\dot{b}_s(\mathbf{k}, \tau) + \varepsilon(\mathbf{k}) b_s(\mathbf{k}, \tau) = \eta_s(\mathbf{k}, \tau)\,,\\
&\dot{b}^*_s(\mathbf{k}, \tau) - \varepsilon(\mathbf{k}) b^*_s(\mathbf{k}, \tau) = -\eta^*_s(\mathbf{k}, \tau)\,,\\
&\dot{d}_s(\mathbf{k}, \tau) + \varepsilon(\mathbf{k}) d_s(\mathbf{k}, \tau) = \overline{\eta}_s(\mathbf{k}, \tau)\,,\\
&\dot{d}^*_s(\mathbf{k}, \tau) - \varepsilon(\mathbf{k}) d^*_s(\mathbf{k}, \tau) = -\overline{\eta}^*_s(\mathbf{k}, \tau)\,.
\end{aligned} \tag{128}
$$

Following the procedure described in (111)–(116), the solutions to the above equations, which satisfies the boundary conditions in (127) and the conditions like those in (113), can be found to be [23–26]

$$
\begin{aligned}
b_s(\mathbf{k}, \tau) &= b_s(\mathbf{k}) \mathrm{e}^{-\varepsilon(\mathbf{k})\tau} + \int_0^\tau \mathrm{d}\tau' \mathrm{e}^{-\varepsilon(\mathbf{k})(\tau-\tau')} \eta_s(\mathbf{k}, \tau')\,,\\
b^*_s(\mathbf{k}, \tau) &= -b^*_s(\mathbf{k}) \mathrm{e}^{\varepsilon(\mathbf{k})(\tau-\beta)} + \int_\tau^\beta \mathrm{d}\tau' \mathrm{e}^{\varepsilon(\mathbf{k})(\tau-\tau')} \eta^*_s(\mathbf{k}, \tau')\,,\\
d_s(\mathbf{k}, \tau) &= d_s(\mathbf{k}) \mathrm{e}^{-\varepsilon(\mathbf{k})\tau} + \int_0^\tau \mathrm{d}\tau' \mathrm{e}^{-\varepsilon(\mathbf{k})(\tau-\tau')} \overline{\eta}_s(\mathbf{k}, \tau')\,,\\
d^*_s(\mathbf{k}, \tau) &= -d^*_s(\mathbf{k}) \mathrm{e}^{\varepsilon(\mathbf{k})(\tau-\beta)} + \int_\tau^\beta \mathrm{d}\tau' \mathrm{e}^{\varepsilon(\mathbf{k})(\tau-\tau')} \overline{\eta}^*_s(\mathbf{k}, \tau')\,.
\end{aligned} \tag{129}
$$

Substituting the above solutions into (102) or (104), we find

$$
\begin{aligned}
&I^0_q\left(b^*_s, b_s, d^*_s, d_s; \eta^*_s, \eta_s, \overline{\eta}^*_s, \overline{\eta}_s\right)\\
&= \int \mathrm{d}^3k \Big\{ -\mathrm{e}^{-\beta\varepsilon(\mathbf{k})} \left[b^*_s(\mathbf{k}) b_s(\mathbf{k}) + d^*_s(\mathbf{k}) d_s(\mathbf{k})\right]\\
&+ \int_0^\beta \mathrm{d}\tau \mathrm{e}^{-\varepsilon(\mathbf{k})\tau} \left[\eta^*_s(\mathbf{k}, \tau) b_s(\mathbf{k}) + \overline{\eta}^*_s(\mathbf{k}, \tau) d_s(\mathbf{k})\right]\\
&- \mathrm{e}^{-\beta\varepsilon(\mathbf{k})} \int_0^\beta \mathrm{d}\tau \mathrm{e}^{\varepsilon(\mathbf{k})\tau} \left[b^*_s(\mathbf{k}) \eta_s(\mathbf{k}, \tau) + d^*_s(\mathbf{k}) \overline{\eta}_s(\mathbf{k}, \tau)\right] \Big\}\\
&+ B\left[\eta^*_s, \eta_s, \overline{\eta}^*_s, \overline{\eta}_s\right]\,,
\end{aligned} \tag{130}
$$

where

$$
\begin{aligned}
&B\left[\eta^*_s, \eta_s, \overline{\eta}^*_s, \overline{\eta}_s\right]\\
&= \frac{1}{2} \int_0^\beta \mathrm{d}\tau_1 \int_0^\beta \mathrm{d}\tau_2 \int \mathrm{d}^3k \Big\{ \theta(\tau_1 - \tau_2) \mathrm{e}^{-\varepsilon(\mathbf{k})(\tau_1-\tau_2)}\\
&\times \left[\eta^*_s(\mathbf{k}, \tau_1) \eta_s(\mathbf{k}, \tau_2) + \overline{\eta}^*_s(\mathbf{k}, \tau_1) \overline{\eta}_s(\mathbf{k}, \tau_2)\right]\\
&+ \theta(\tau_2 - \tau_1) \mathrm{e}^{-\varepsilon(\mathbf{k})(\tau_2-\tau_1)}\\
&\times \left[\eta^*_s(\mathbf{k}, \tau_2) \eta_s(\mathbf{k}, \tau_1) + \overline{\eta}^*_s(\mathbf{k}, \tau_2) \overline{\eta}_s(\mathbf{k}, \tau_1)\right] \Big\}\,.
\end{aligned} \tag{131}
$$

On inserting (130) into (126), we have

$$
Z^0_q[\eta] = A\left[\eta^*_s, \eta_s, \overline{\eta}^*_s, \overline{\eta}_s\right] \mathrm{e}^{B\left[\eta^*_s, \eta_s, \overline{\eta}^*_s, \overline{\eta}_s\right]}\,, \tag{132}
$$

where

$$
\begin{aligned}
&A\left[\eta^*_s, \eta_s, \overline{\eta}^*_s, \overline{\eta}_s\right]\\
&= \int D(b^* b) \exp\Big\{ -\int \mathrm{d}^3k \Big[b^*_s(\mathbf{k}) b_s(\mathbf{k}) (1 + \mathrm{e}^{-\beta\varepsilon(\mathbf{k})})\\
&+ \mathrm{e}^{-\beta\varepsilon(\mathbf{k})} b^*_s(\mathbf{k}) \int_0^\beta \mathrm{d}\tau \mathrm{e}^{\varepsilon(\mathbf{k})\tau} \eta_s(\mathbf{k}, \tau)\\
&- \int_0^\beta \mathrm{d}\tau \mathrm{e}^{-\varepsilon(\mathbf{k})\tau} \eta^*_s(\mathbf{k}, \tau) b_s(\mathbf{k}) \Big] \Big\}\\
&\times \int D(d^* d) \exp\Big\{ -\int \mathrm{d}^3k \Big[d^*_s(\mathbf{k}) d_s(\mathbf{k}) (1 + \mathrm{e}^{-\beta\varepsilon(\mathbf{k})})\\
&+ \mathrm{e}^{-\beta\varepsilon(\mathbf{k})} d^*_s(\mathbf{k}) \int_0^\beta \mathrm{d}\tau \mathrm{e}^{\varepsilon(\mathbf{k})\tau} \overline{\eta}_s(\mathbf{k}, \tau)\\
&- \int_0^\beta \mathrm{d}\tau \mathrm{e}^{-\varepsilon(\mathbf{k})\tau} \overline{\eta}^*_s(\mathbf{k}, \tau) d_s(\mathbf{k}) \Big] \Big\}\,,
\end{aligned} \tag{133}
$$

where the fact that the two integrals over $\{b^*, b\}$ and $\{d^*, d\}$ commute with each other has been noted. Obviously, each of the above integrals can easily be calculated by applying the integration formulas shown in (39). The result is

$$
\begin{aligned}
&A\left[\eta^*_s, \eta_s, \overline{\eta}^*_s, \overline{\eta}_s\right]\\
&= Z^0_q \exp\Big\{ -\frac{1}{2} \int_0^\beta \mathrm{d}\tau_1 \int_0^\beta \mathrm{d}\tau_2 \int \mathrm{d}^3k (1 + \mathrm{e}^{\beta\varepsilon(\mathbf{k})})^{-1}\\
&\times \Big\{ \mathrm{e}^{-\varepsilon(\mathbf{k})(\tau_1-\tau_2)} \left[\eta^*_s(\mathbf{k}, \tau_1) \eta_s(\mathbf{k}, \tau_2) + \overline{\eta}^*_s(\mathbf{k}, \tau_1) \overline{\eta}_s(\mathbf{k}, \tau_2)\right]\\
&+ \mathrm{e}^{-\varepsilon(\mathbf{k})(\tau_2-\tau_1)}\\
&\times \left[\eta^*_s(\mathbf{k}, \tau_2) \eta_s(\mathbf{k}, \tau_1) + \overline{\eta}^*_s(\mathbf{k}, \tau_2) \overline{\eta}_s(\mathbf{k}, \tau_1)\right] \Big\} \Big\}\,,
\end{aligned} \tag{134}
$$

where

$$
Z^0_q = \prod_{\mathbf{k}s} \left[1 + \mathrm{e}^{-\beta\varepsilon(\mathbf{k})}\right]^2 \tag{135}
$$

which just is the partition function contributed from free quarks and antiquarks [21–23, 29]. It is noted that the two terms in the exponent of (134) are equal to one another as seen from the interchange of the integration variables τ_1 and τ_2. After (131) and (134) are substituted in (132), we get

$$
\begin{aligned}
Z^0_q[\eta] &= Z^0_q \exp\Big\{ \int_0^\beta \mathrm{d}\tau_1 \int_0^\beta \mathrm{d}\tau_2 \int \mathrm{d}^3k\\
&\times \Big\{ \Big[\theta(\tau_1 - \tau_2) - (1 + \mathrm{e}^{\beta\varepsilon(\mathbf{k})})^{-1}\Big] \mathrm{e}^{-\varepsilon(\mathbf{k})(\tau_1-\tau_2)}\\
&\times \left[\eta^*_s(\mathbf{k}, \tau_1) \eta_s(\mathbf{k}, \tau_2) + \overline{\eta}^*_s(\mathbf{k}, \tau_1) \overline{\eta}_s(\mathbf{k}, \tau_2)\right]\\
&+ \Big[\theta(\tau_2 - \tau_1) - (1 + \mathrm{e}^{\beta\varepsilon(\mathbf{k})})^{-1}\Big] \mathrm{e}^{-\varepsilon(\mathbf{k})(\tau_2-\tau_1)}\\
&\times \left[\eta^*_s(\mathbf{k}, \tau_2) \eta_s(\mathbf{k}, \tau_1) + \overline{\eta}^*_s(\mathbf{k}, \tau_2) \overline{\eta}_s(\mathbf{k}, \tau_1)\right] \Big\} \Big\}\,.
\end{aligned} \tag{136}
$$

When we interchange the variables τ_1 and τ_2 and set $\mathbf{k} \to -\mathbf{k}$ in the second term of the above integrals and notice the relation

$$\overline{\eta}_s^*(\mathbf{k},\tau_2)\overline{\eta}_s(\mathbf{k},\tau_1) = \eta_s^*(-\mathbf{k},\tau_1)\eta_s(-\mathbf{k},\tau_2)\,, \qquad (137)$$

which will be proved in the appendix, the functional $Z_q^0[\eta]$ will eventually be represented as

$$Z_q^0[\eta] = Z_q^0 \exp\left\{ \int_0^\beta \mathrm{d}\tau_1 \int_0^\beta \mathrm{d}\tau_2 \int \mathrm{d}^3k \times \left[\eta_s^*(\mathbf{k},\tau_1)\Delta_q^{ss'}(\mathbf{k},\tau_1-\tau_2)\eta_{s'}(\mathbf{k},\tau_2) + \overline{\eta}_s^*(\mathbf{k},\tau_1)\Delta_q^{ss'}(\mathbf{k},\tau_1-\tau_2)\overline{\eta}_{s'}(\mathbf{k},\tau_2)\right]\right\}, \qquad (138)$$

where

$$\Delta_q^{ss'}(\mathbf{k},\tau_1-\tau_2) = \delta^{ss'}\Delta_q(\mathbf{k},\tau_1-\tau_2)\,; \qquad (139)$$

in which

$$\Delta_q(\mathbf{k},\tau_1-\tau_2) = \frac{1}{2}\left[\overline{n}_f(\mathbf{k})\mathrm{e}^{-\varepsilon(\mathbf{k})|\tau_1-\tau_2|} - n_f(\mathbf{k})\mathrm{e}^{\varepsilon(\mathbf{k})|\tau_1-\tau_2|}\right] \qquad (140)$$

is the free quark (antiquark) propagator. In the above,

$$\overline{n}_f(\mathbf{k}) = \left(1+\mathrm{e}^{-\beta\varepsilon(\mathbf{k})}\right)^{-1}, \quad n_f(\mathbf{k}) = \left(1+\mathrm{e}^{\beta\varepsilon(\mathbf{k})}\right)^{-1} \qquad (141)$$

are the fermion distribution functions [21–23, 29].

Finally, let us calculate the generating functional $Z_c^0[\zeta]$. From the stationary condition $\delta I_c(c_a^*, c_a, \overline{c}_a^*, \overline{c}_a; \zeta_a^*, \zeta_a, \overline{\zeta}_a^*, \overline{\zeta}_a) = 0$ and the boundary conditions

$$\begin{aligned} &\overline{c}_a^*(\mathbf{k},\beta) = \overline{c}_a^*(\mathbf{k})\,, \quad c_a^*(\mathbf{k},\beta) = c_a^*(\mathbf{k})\,,\\ &\overline{c}_a(\mathbf{k},0) = \overline{c}_a(\mathbf{k})\,, \quad c_a(\mathbf{k},0) = c_a(\mathbf{k})\,, \end{aligned} \qquad (142)$$

which are the same as those for scalar fields and different from fermion fields [21], it is easy to derive from (106) or (108) the following equations of motion:

$$\begin{aligned} &\dot{c}_a(\mathbf{k},\tau) + \omega(\mathbf{k})c_a(\mathbf{k},\tau) = -\zeta_a(\mathbf{k},\tau),\\ &\dot{c}_a^*(\mathbf{k},\tau) - \omega(\mathbf{k})c_a^*(\mathbf{k},\tau) = \zeta_a^*(\mathbf{k},\tau),\\ &\dot{\overline{c}}_a(\mathbf{k},\tau) + \omega(\mathbf{k})\overline{c}_a(\mathbf{k},\tau) = \overline{\zeta}_a(\mathbf{k},\tau),\\ &\dot{\overline{c}}_a^*(\mathbf{k},\tau) - \omega(\mathbf{k})\overline{c}_a^*(\mathbf{k},\tau) = -\overline{\zeta}_a^*(\mathbf{k},\tau). \end{aligned} \qquad (143)$$

By the same procedure as stated in (111)–(116), the solutions to the above equations can be found to be

$$\begin{aligned} c_a(\mathbf{k},\tau) &= c_a(\mathbf{k})\mathrm{e}^{-\omega(\mathbf{k})\tau} - \int_0^\tau \mathrm{d}\tau'\mathrm{e}^{-\omega(\mathbf{k})(\tau-\tau')}\zeta_a(\mathbf{k},\tau')\,,\\ c_a^*(\mathbf{k},\tau) &= c_a^*(\mathbf{k})\mathrm{e}^{\omega(\mathbf{k})(\tau-\beta)} - \int_\tau^\beta \mathrm{d}\tau'\mathrm{e}^{\omega(\mathbf{k})(\tau-\tau')}\zeta_a^*(\mathbf{k},\tau')\,,\\ \overline{c}_a(\mathbf{k},\tau) &= \overline{c}_a(\mathbf{k})\mathrm{e}^{-\omega(\mathbf{k})\tau} + \int_0^\tau \mathrm{d}\tau'\mathrm{e}^{-\omega(\mathbf{k})(\tau-\tau')}\overline{\zeta}_a(\mathbf{k},\tau')\,,\\ \overline{c}_a^*(\mathbf{k},\tau) &= \overline{c}_a^*(\mathbf{k})\mathrm{e}^{\omega(\mathbf{k})(\tau-\beta)} + \int_\tau^\beta \mathrm{d}\tau'\mathrm{e}^{\omega(\mathbf{k})(\tau-\tau')}\overline{\zeta}_a^*(\mathbf{k},\tau')\,. \end{aligned} \qquad (144)$$

Upon substituting the above solutions into (106) or (108), we find

$$\begin{aligned} &I_c^0\left(c_a^*, c_a, \overline{c}_a^*, \overline{c}_a; \zeta_a^*, \zeta_a, \overline{\zeta}_a^*, \overline{\zeta}_a\right)\\ &= \int \mathrm{d}^3k\left\{\mathrm{e}^{-\beta\omega(\mathbf{k})}\left[\overline{c}_a^*(\mathbf{k})\overline{c}_a(\mathbf{k}) - c_a^*(\mathbf{k})c_a(\mathbf{k})\right]\right.\\ &\quad + \mathrm{e}^{-\beta\omega(\mathbf{k})}\int_0^\beta \mathrm{d}\tau\mathrm{e}^{\omega(\mathbf{k})\tau}\left[c_a^*(\mathbf{k})\zeta_a(\mathbf{k},\tau) + \overline{c}_a^*(\mathbf{k})\overline{\zeta}_a(\mathbf{k},\tau)\right]\\ &\quad \left. - \int_0^\beta \mathrm{d}\tau\mathrm{e}^{-\omega(\mathbf{k})\tau}\left[\zeta_a^*(\mathbf{k},\tau)c_a(\mathbf{k}) + \overline{\zeta}_a^*(\mathbf{k},\tau)\overline{c}_a(\mathbf{k})\right]\right\}\\ &\quad - \int_0^\beta \mathrm{d}\tau_1\int_0^\beta \mathrm{d}\tau_2\int \mathrm{d}^3k\theta(\tau_1-\tau_2)\mathrm{e}^{-\omega(\mathbf{k})(\tau_1-\tau_2)}\\ &\quad \times\left[\zeta_a^*(\mathbf{k},\tau_1)\zeta_a(\mathbf{k},\tau_2) - \overline{\zeta}_a^*(\mathbf{k},\tau_1)\overline{\zeta}_a(\mathbf{k},\tau_2)\right]. \end{aligned} \qquad (145)$$

On inserting the above expression into the following integral given by the stationary-phase method:

$$Z_c^0[\zeta] = \int D(\overline{c}^*\overline{c}cc^*) \times \exp\left\{-\int \mathrm{d}^3k\left[\overline{c}_a^*(\mathbf{k})\overline{c}_a(\mathbf{k}) - c_a^*(\mathbf{k})c_a(\mathbf{k})\right] + I_c^0\left(c_a^*, c_a, \overline{c}_a^*, \overline{c}_a; \zeta_a^*, \zeta_a, \overline{\zeta}_a^*, \overline{\zeta}_a\right)\right\}, \qquad (146)$$

and applying the integration formulas in (39), one can get

$$Z_c^0[\zeta] = Z_c^0 \exp\left\{\int_0^\beta \mathrm{d}\tau_1\int_0^\beta \mathrm{d}\tau_2\int \mathrm{d}^3k \times\left[\theta(\tau_1-\tau_2) - (1-\mathrm{e}^{\beta\omega(\mathbf{k})})^{-1}\right]\mathrm{e}^{-\omega(\mathbf{k})(\tau_1-\tau_2)} \times\left[\overline{\zeta}_a^*(\mathbf{k},\tau_1)\overline{\zeta}_a(\mathbf{k},\tau_2) - \zeta_a^*(\mathbf{k},\tau_1)\zeta_a(\mathbf{k},\tau_2)\right]\right\}, \qquad (147)$$

where

$$Z_c^0 == \prod_{\mathbf{k}a}\left[1-\mathrm{e}^{-\beta\omega(\mathbf{k})}\right]^2 \qquad (148)$$

is just the partition function arising from the free ghost particles which plays the role of cancelling out the unphysical contribution contained in (120). If we change the integration variables in (147) and consider the relations

$$\zeta_a^*(\mathbf{k},\tau) = -\overline{\zeta}_a(-\mathbf{k},\tau),\ \zeta_a(\mathbf{k},\tau) = -\overline{\zeta}_a^*(-\mathbf{k},\tau)\,, \qquad (149)$$

which will be interpreted in the appendix, (147) may be recast in the form

$$Z_c^0[\zeta]=Z_c^0\exp\left\{\int_0^\beta \mathrm{d}\tau_1\int_0^\beta \mathrm{d}\tau_2\int \mathrm{d}^3k \times\left[\overline{\zeta}_a^*(\mathbf{k},\tau_1)\Delta_c^{aa'}(\mathbf{k},\tau_1-\tau_2)\overline{\zeta}_{a'}(\mathbf{k},\tau_2) -\zeta_a^*(\mathbf{k},\tau_1)\Delta_c^{aa'}(\mathbf{k},\tau_1-\tau_2)\zeta_{a'}(\mathbf{k},\tau_2)\right]\right\}, \quad (150)$$

where

$$\Delta_c^{aa'}(\mathbf{k},\tau_1-\tau_2)=\delta^{aa'}\Delta_g(\mathbf{k},\tau_1-\tau_2)\,; \quad (151)$$

here $\Delta_g(\mathbf{k},\tau_1-\tau_2)$ was written in (124).

Up to the present, the perturbative expansion of the thermal QCD generating functional in the coherent-state representation has exactly been given by the combination of (95), (96), (119), (138) and (150). In the derivation of the perturbation expansion, as one has seen, to obtain the final expressions of the propagators shown in (124), (140) and (151), it is necessary to use the functional properties and relations for the external sources as denoted in (123), (137) and (149). As a result of the derivation of the generating functional, the partition function for the free system has simultaneously been given by the combination of (120), (135) and (148). The partition function for the interacting system can be calculated in the way as shown in (45). Here it should be noted that the differential $\delta/\delta j(\tau)$ in (95) represent the collection of the differentials $\delta/\delta\xi_\lambda^{a*}(\mathbf{k},\tau)$, $\delta/\delta\xi_\lambda^{a}(\mathbf{k},\tau)$, $-\delta/\delta\eta_s(\mathbf{k},\tau)$, $\delta/\delta\eta_s^*(\mathbf{k},\tau)$, $-\delta/\delta\overline{\eta}_s(\mathbf{k},\tau)$, $\delta/\delta\overline{\eta}_s^*(\mathbf{k},\tau)$, $-\delta/\delta\zeta_a(\mathbf{k},\tau)$, $\delta/\delta\zeta_a^*(\mathbf{k},\tau)$, $-\delta/\delta\overline{\zeta}_a(\mathbf{k},\tau)$ and $\delta/\delta\overline{\zeta}_a^*(\mathbf{k},\tau)$. Ordinarily, the generating functional of thermal QCD represented in the position space is used in the literature. In the appendix, it will be shown that this generating functional can readily be derived from the generating functional described in this section.

5 Relativistic equation for $q\overline{q}$ bound states

With the generating functional given in the preceding section, we are ready to derive the relativistic equation for $q\overline{q}$ bound states at finite temperature. It is well-known that a bound state exists in the space-like Minkowski space in which there always is an equal-time Lorentz frame. Since in the equal-time frame, the relativistic equation is reduced to a three-dimensional one without loss of any rigor, in this section we only pay attention to the three-dimensional equation which may be derived from the equations of motion satisfied by the following $q\overline{q}$ two-“time” (temperature) four-point Green function [15–17]:

$$\begin{aligned}&\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)\\&=\mathrm{Tr}\left\{\mathrm{e}^{\beta(\Omega-\widehat{K})}T\left\{N\left[\widehat{b}_\alpha(\tau_1)\widehat{b}_\beta^+(\tau_1)\right]N\left[\widehat{b}_\gamma(\tau_2)\widehat{b}_\delta^+(\tau_2)\right]\right\}\right\}\\&\equiv\left\langle T\left\{N\left[\widehat{b}_\alpha(\tau_1)\widehat{b}_\beta^+(\tau_1)\right]N\left[\widehat{b}_\gamma(\tau_2)\widehat{b}_\delta^+(\tau_2)\right]\right\}\right\rangle_\beta,\end{aligned} \quad (152)$$

where the symbol $\langle\rangle_\beta$ represents the statistical average and N symbolizes the normal product whose definition can be given from the corresponding definition at zero temperature by replacing the vacuum average with the statistical average [17]

$$N\left[\widehat{b}_\alpha(\tau_1)\widehat{b}_\beta^+(\tau_2)\right]=T\left[\widehat{b}_\alpha(\tau_1)\widehat{b}_\beta^+(\tau_2)\right]-S_{\alpha\beta}(\tau_1-\tau_2)\,, \quad (153)$$

where

$$S_{\alpha\beta}(\tau_1-\tau_2)=\left\langle T\left[\widehat{b}_\alpha(\tau_1)\widehat{b}_\beta^+(\tau_2)\right]\right\rangle_\beta \quad (154)$$

is the quark or antiquark thermal propagator. The normal product in (152) plays the role of excluding the contraction between the quark and the antiquark operators from the Green function when the quark and antiquark are of the same flavor. Physically, this avoids the $q\overline{q}$ annihilation that would break stability of a bound state. Substituting (153) in (152), we have

$$\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)=G(\alpha\beta;\gamma\delta;\tau_1-\tau_2)-S_{\alpha\beta}S_{\gamma\delta}\,, \quad (155)$$

where

$$G(\alpha\beta;\gamma\delta;\tau_1-\tau_2)=\left\langle T\left\{\widehat{b}_\alpha(\tau_1)\widehat{b}_\beta^+(\tau_1)\widehat{b}_\gamma(\tau_2)\widehat{b}_\delta^+(\tau_2)\right\}\right\rangle_\beta \quad (156)$$

is the ordinary Green function, and $S_{\alpha\beta}$ and $S_{\gamma\delta}$ are the equal-time quark (antiquark) propagators. Obviously, in order to derive the equation of motion satisfied by the Green function $\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)$, we need first to derive the equation of motion for the Green function $G(\alpha\beta;\gamma\delta;\tau_1-\tau_2)$.

Let us start with the generating functional in (91). As shown in Sect. 4, by partial integration of the second term on the right hand side of (89), the generating functional may be written in the form

$$Z[j]=\int D(q^*\cdot q)\mathrm{e}^{-q^*\cdot q}\int\mathfrak{D}(q^*\cdot q)\times\exp\left\{q^*(\beta)\cdot q(\beta)-S_{\mathrm{E}}+\int_0^\beta \mathrm{d}\tau j^*(\tau)\cdot q(\tau)\right\}, \quad (157)$$

where

$$S_{\mathrm{E}}=\int_0^\beta \mathrm{d}\tau\Big\{\sum_\alpha q_\alpha^*(\tau)\circ\dot{q}_\alpha(\tau)+H(\tau)\Big\}\,; \quad (158)$$

here $H(\tau)$ was given in (78) and (79). First, we derive an equation of motion describing the variation of the $q\overline{q}$ four-point Green function with the “time” variable τ_1. For this purpose, let us differentiate the generating functional in (157) with respect to $b_\alpha^*(\tau_1)$. Considering that the generating functional is independent of $b_\alpha^*(\tau_1)$ and noticing the expressions given in (158), (78), (79) and (93), one may obtain

$$\frac{\delta Z[j]}{\delta b_\alpha^*(\tau_1)} = \int D(q^*q)\mathrm{e}^{-q^*q}\int \mathfrak{D}(q^*q)\Big[-\dot{b}_\alpha(\tau_1)-\varepsilon_\alpha\theta_\alpha b_\alpha(\tau_1) - \sum_{\rho\lambda}A_{\alpha\rho\lambda}b_\rho(\tau_1)a_\lambda(\tau_1)+\theta_\alpha\eta_\alpha(\tau_1)\Big] \times\exp\left\{q^*(\beta)q(\beta)-S_{\mathrm{E}}-\int_0^\beta \mathrm{d}\tau j^*(\tau)q(\tau)\right\}=0. \tag{159}$$

When the $b_\alpha(\tau_1)$ and $a_\lambda(\tau_1)$ in the above are replaced by the functional derivatives $\theta_\alpha\delta/\delta\eta_\alpha^*(\tau_1)$ and $\delta/\delta j_\lambda^*(\tau_1)$ respectively and multiplying both sides of (159) with θ_α, the above equation can be written as

$$\left\{\frac{\mathrm{d}}{\mathrm{d}\tau_1}\frac{\delta}{\delta\eta_\alpha^*(\tau_1)}+\theta_\alpha\varepsilon_\alpha\frac{\delta}{\delta\eta_\alpha^*(\tau_1)} + \sum_{\rho\lambda}\theta_\alpha\theta_\rho A(\alpha\rho\lambda)\frac{\delta^2}{\delta\eta_\rho^*(\tau_1)\delta j_\lambda^*(\tau_1)}-\eta_\alpha(\tau_1)\right\}Z[j]=0. \tag{160}$$

Then we differentiate the above equation with respect to the sources $\eta_\beta((\tau_1)$, giving

$$\left\{\left(\frac{\mathrm{d}}{\mathrm{d}\tau_1}\frac{\delta}{\delta\eta_\alpha^*(\tau_1)}\right)\frac{\delta}{\delta\eta_\beta(\tau_1)}+\theta_\alpha\varepsilon_\alpha\frac{\delta^2}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)} + \sum_{\rho\lambda}\theta_\alpha\theta_\rho A(\alpha\rho\lambda)\frac{\delta^3}{\delta\eta_\rho^*(\tau_1)\delta\eta_\beta(\tau_1)\delta j_\lambda^*(\tau_1)} + \delta_{\alpha\beta}-\eta_\alpha(\tau_1)\frac{\delta}{\delta\eta_\beta(\tau_1)}\right\}Z[j]=0. \tag{161}$$

Furthermore, successive differentiations of (161) with respect sources $\eta_\gamma^*(\tau_2)$ and $\eta_\delta(\tau_2)$ yield

$$\left\{\left(\frac{\mathrm{d}}{\mathrm{d}\tau_1}\frac{\delta}{\delta\eta_\alpha^*(\tau_1)}\right)\frac{\delta^3}{\delta\eta_\beta(\tau_1)\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)} +\theta_\alpha\varepsilon_\alpha\frac{\delta^4}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)} +\sum_{\lambda\sigma}\theta_\alpha\theta_\rho A(\alpha\rho\lambda)\frac{\delta^5}{\delta\eta_\rho^*(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)\delta j_\lambda^*(\tau_1)} +\delta_{\alpha\beta}\frac{\delta^2}{\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)}-\delta_{\alpha\delta}\delta(\tau_1-\tau_2)\frac{\delta^2}{\delta\eta_\gamma^*(\tau_2)\delta\eta_\beta(\tau_1)} -\eta_\alpha(\tau_1)\frac{\delta^3}{\delta\eta_\beta(\tau_1)\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)}\right\}Z[j]=0. \tag{162}$$

Similarly, when differentiating (157) with respect $b_\beta(\tau_1)$, one may obtain

$$\left\{\frac{\mathrm{d}}{\mathrm{d}\tau_1}\frac{\delta}{\delta\eta_\beta(\tau_1)}-\theta_\beta\varepsilon_\beta\frac{\delta}{\delta\eta_\beta(\tau_1)} -\sum_{\sigma\lambda}\theta_\sigma\theta_\beta A(\sigma\beta\lambda)\frac{\delta^2}{\delta\eta_\sigma(\tau_1)\delta j_\lambda^*(\tau_1)}-\eta_\beta^*(\tau_1)\right\}Z[j]=0, \tag{163}$$

Subsequently, on differentiating the above equation with respect to $\eta_\alpha^*(\tau_1)$, we get

$$\left\{\frac{\delta}{\delta\eta_\alpha^*(\tau_1)}\left(\frac{\mathrm{d}}{\mathrm{d}\tau_1}\frac{\delta}{\delta\eta_\beta((\tau_1)}\right)-\theta_\beta\varepsilon_\beta\frac{\delta^2}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)} -\sum_{\sigma\lambda}\theta_\beta\theta_\sigma A(\sigma\beta\lambda)\frac{\delta^3}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\sigma(\tau_1)\delta j_\lambda^*(\tau_1)} -\delta_{\alpha\beta}+\eta_\beta^*(\tau_1)\frac{\delta}{\delta\eta_\alpha^*(\tau_1)}\right\}Z[j]=0. \tag{164}$$

Finally, successive differentiations of the above equation with respect to the sources $\eta_\gamma^*(\tau_2)$ and $\eta_\delta(\tau_2)$ give rise to

$$\left\{\frac{\delta}{\delta\eta_\alpha^*(\tau_1)}\left(\frac{\mathrm{d}}{\mathrm{d}\tau_1}\frac{\delta}{\delta\eta_\beta(\tau_1)}\right)\frac{\delta^2}{\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)} -\theta_\beta\varepsilon_\beta\frac{\delta^4}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)} -\sum_{\lambda\sigma}\theta_\beta\theta_\sigma A(\sigma\beta\lambda)\frac{\delta^5}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\sigma(\tau_1)\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)\delta j_\lambda^*(\tau_1)} -\delta_{\alpha\beta}\frac{\delta^2}{\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)}+\delta_{\beta\gamma}\delta(\tau_1-\tau_2)\frac{\delta^2}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\delta(\tau_2)} +\eta_\beta^*(\tau_1)\frac{\delta^3}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)}\right\}Z[j]=0. \tag{165}$$

Adding (161) to (164), then multiplying both sides of the equation thus obtained with $-\theta_\alpha\theta_\beta$ and finally setting the external sources $\eta_\alpha^*=\eta_\beta=0$, but keeping the gluon source $j_\lambda\neq 0$, we get

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau_1}+\theta_\alpha\varepsilon_\alpha-\theta_\beta\varepsilon_\beta\right)S_{\alpha\beta}^{j_\lambda} +\sum_{\rho\sigma\lambda}[A(\alpha\rho\lambda)\delta_{\beta\sigma}-A(\sigma\beta\lambda)\delta_{\alpha\rho}]\frac{\delta}{\delta j_\lambda^*(\tau_1)}S_{\rho\sigma}^{j_\lambda}=0\,, \tag{166}$$

where

$$S_{\alpha\beta}^{j_\lambda}=-\frac{1}{Z}\theta_\alpha\theta_\beta\frac{\delta^2 Z[j]}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)}\Big|_{\eta_\alpha^*=\eta_\beta=0} \tag{167}$$

is the quark (antiquark) equal-time propagator in the presence of source j_λ. If we define

$$H(\alpha\beta;\rho\sigma;\tau_1)^{j_\lambda} = \left(\frac{\mathrm{d}}{\mathrm{d}\tau_1} + \theta_\alpha\varepsilon_\alpha - \theta_\beta\varepsilon_\beta\right)\delta_{\alpha\rho}\delta_{\beta\sigma} + \sum_\lambda f(\alpha\beta;\rho\sigma\lambda)\frac{\delta}{\delta j_\lambda^*(\tau_1)}\,, \qquad (168)$$

where

$$f(\alpha\beta;\rho\sigma\lambda) = A(\alpha\rho\lambda)\delta_{\beta\sigma} - A(\sigma\beta\lambda)\delta_{\alpha\rho}\,, \qquad (169)$$

(166) can simply be represented as

$$\sum_{\rho\sigma} H(\alpha\beta;\rho\sigma;\tau_1)^{j_\lambda} S_{\rho\sigma}^{j_\lambda} = 0\,. \qquad (170)$$

When summing up both equations in (162) and (165), then multiplying the equation thus obtained with $\theta_\alpha\theta_\beta\theta_\gamma\theta_\delta$ and finally setting all the sources but j_λ to be zero, one may get

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau_1} + \theta_\alpha\varepsilon_\alpha - \theta_\beta\varepsilon_\beta\right) G(\alpha\beta;\gamma\delta;\tau_1-\tau_2)^{j_\lambda} + \sum_{\rho\sigma\lambda} f(\alpha\beta;\rho\sigma\lambda)\frac{\delta}{\delta j_\lambda^*(\tau_1)} G(\rho\sigma;\gamma\delta;\tau_1-\tau_2)^{j_\lambda} = \delta(\tau_1-\tau_2)\left[\delta_{\beta\gamma}S_{\alpha\delta}(\tau_1-\tau_2)^{j_\lambda} - \delta_{\alpha\delta}S_{\gamma\beta}(\tau_2-\tau_1)^{j_\lambda}\right]\,, \qquad (171)$$

where

$$G(\alpha\beta;\gamma\delta;\tau_1-\tau_2)^{j_\lambda} = \frac{1}{Z}\theta_\alpha\theta_\beta\theta_\gamma\theta_\delta \frac{\delta^4 Z[j]}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)}\Big|_{\eta^*=\eta=0} \qquad (172)$$

and

$$S_{\alpha\beta}(\tau_1-\tau_2)^{j_\lambda} = -\frac{1}{Z}\theta_\alpha\theta_\beta \frac{\delta^2 Z[j]}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_2)}\Big|_{\eta^*=\eta=0} \qquad (173)$$

are respectively the $q\overline{q}$ two-"time" four-point thermal Green function and the quark or antiquark thermal propagator in the presence of the source j_λ. When the source j_λ is turned off, (172) and (173) will respectively go over to the Green function in (156) and the propagator in (154). It is noted that due to the restriction of the delta function, the propagators in (171) are actually "time"-independent. With the definition in (168), (171) may be represented as

$$\sum_{\rho\sigma} H(\alpha\beta;\rho\sigma;\tau_1)^{j_\lambda} G(\rho\sigma;\gamma\delta;\tau_1-\tau_2)^{j_\lambda} = -\delta(\tau_1-\tau_2)S(\alpha\beta;\gamma\delta)^{j_\lambda}\,, \qquad (174)$$

where

$$S(\alpha\beta;\gamma\delta)^{j_\lambda} = \delta_{\alpha\delta}S_{\gamma\beta}^{j_\lambda} - \delta_{\beta\gamma}S_{\alpha\delta}^{j_\lambda}\,. \qquad (175)$$

Acting on both sides of (155) with the operator $H(\alpha\beta;\rho\sigma;\tau_1)^{j_\lambda}$ and using the equations in (170) and (174), we find

$$\sum_{\rho\sigma} H(\alpha\beta;\rho\sigma;\tau_1)^{j_\lambda}\mathcal{G}(\rho\sigma;\gamma\delta;\tau_1-\tau_2)^{j_\lambda} = \sum_{\rho\sigma} H(\alpha\beta;\rho\sigma;\tau_1)^{j_\lambda} G(\rho\sigma;\gamma\delta;\tau_1-\tau_2)^{j_\lambda} = -\delta(\tau_1-\tau_2)S(\alpha\beta;\gamma\delta)^{j_\lambda}\,. \qquad (176)$$

This indicates that the equation of motion satisfied by the Green function $\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)$ formally is the same as the one shown in (171). Therefore, in the case that the source j_λ vanishes, we can write

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau_1} + \theta_\alpha\varepsilon_\alpha - \theta_\beta\varepsilon_\beta\right)\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2) = -\delta(\tau_1-\tau_2)S(\alpha\beta;\gamma\delta) - \sum_{\rho\sigma\lambda} f(\alpha\beta;\rho\sigma\lambda)\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\tau_1-\tau_2)\,, \qquad (177)$$

where

$$\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\tau_1-\tau_2) = \frac{\delta}{\delta j_\lambda^*(\tau_1)}\mathcal{G}(\rho\sigma;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}\Big|_{j_\lambda=0} = \left\langle T\left\{N\left[\widehat{b}_\rho(\tau_1)\widehat{b}_\sigma^+(\tau_1)\widehat{a}_\lambda(\tau_1)\right]N\left[\widehat{b}_\gamma(\tau_2)\widehat{b}_\delta^+(\tau_2)\right]\right\}\right\rangle_\beta \qquad (178)$$

and

$$S(\alpha\beta;\gamma\delta) = \delta_{\alpha\delta}S_{\gamma\beta} - \delta_{\beta\gamma}S_{\alpha\delta} = -\left\langle\left[\widehat{b}_\alpha\widehat{b}_\beta^+,\widehat{b}_\gamma\widehat{b}_\delta^+\right]_-\right\rangle_\beta\,. \qquad (179)$$

It is noted here that similar to the definition in (153), the normal product $N[\widehat{b}_\rho(\tau_1)\widehat{b}_\sigma^+(\tau_1)\widehat{a}_\lambda(\tau_1)]$ in (178) is defined as

$$N\left[\widehat{b}_\rho(\tau_1)\widehat{b}_\sigma^+(\tau_1)\widehat{a}_\lambda(\tau_1)\right] = T\left[\widehat{b}_\rho(\tau_1)\widehat{b}_\sigma^+(\tau_1)\widehat{a}_\lambda(\tau_1)\right] - \Lambda(\rho\sigma\lambda)\,, \qquad (180)$$

where

$$\Lambda(\rho\sigma\lambda) = \left\langle T\left[\widehat{b}_\rho(\tau_1)\widehat{b}_\sigma^+(\tau_1)\widehat{a}_\lambda(\tau_1)\right]\right\rangle_\beta\,. \qquad (181)$$

Substituting (153) and (180) into (178), we have

$$\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\tau_1-\tau_2) = G(\rho\sigma\lambda;\gamma\delta;\tau_1-\tau_2) - \Lambda(\rho\sigma\lambda)S_{\gamma\delta}\,, \qquad (182)$$

where

$$G(\rho\sigma\lambda;\gamma\delta;\tau_1-\tau_2) = \left\langle T\left\{\widehat{b}_\rho(\tau_1)\widehat{b}_\sigma^+(\tau_1)\widehat{a}_\lambda(\tau_1)\widehat{b}_\gamma(\tau_2)\widehat{b}_\delta^+(\tau_2)\right\}\right\rangle_\beta \qquad (183)$$

is the ordinary five-point thermal Green function including a gluon operator in it.

By the argument as mentioned in the appendix (see (A.9) or (A.18)), it is easy to prove that the Green functions $\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)$ and $\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\tau_1-\tau_2)$ are periodic. Therefore, we have the following Fourier expansions:

$$\mathcal{G}(\alpha\beta;\gamma\delta;\tau)=\frac{1}{\beta}\sum_n \mathcal{G}(\alpha\beta;\gamma\delta;\omega_n)\mathrm{e}^{-\mathrm{i}\omega_n\tau}\,,$$
$$\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\tau)=\frac{1}{\beta}\sum_n \mathcal{G}(\rho\sigma\lambda;\gamma\delta;\omega_n)\mathrm{e}^{-\mathrm{i}\omega_n\tau}\,,\quad (184)$$

where $\tau=\tau_1-\tau_2$ and $\omega_n=\frac{2\pi n}{\beta}$. Upon inserting (184) into (177) and performing the integration $\frac{1}{2}\int_{-\beta}^{\beta}\mathrm{d}\tau\mathrm{e}^{\mathrm{i}\omega_n\tau}$, we arrive at

$$(\mathrm{i}\omega_n-\theta_\alpha\varepsilon_\alpha+\theta_\beta\varepsilon_\beta)\mathcal{G}(\alpha\beta;\gamma\delta;\omega_n)$$
$$=-S(\alpha\beta;\gamma\delta)+\sum_{\rho\sigma\lambda}f(\alpha\beta;\rho\sigma\lambda)\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\omega_n)\,. \quad (185)$$

It is well-known that the Green function $\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\omega_n)$ is B-S (two-particle) reducible [15–17]. Therefore, we can write

$$\sum_{\lambda\tau\rho}f(\alpha\beta;\rho\sigma\lambda)\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\omega_n)$$
$$=\sum_{\mu\nu}K(\alpha\beta;\mu\nu;\omega_n)\mathcal{G}(\mu\nu;\gamma\delta;\omega_n)\,, \quad (186)$$

where $K(\alpha\beta;\mu\nu;\omega_n)$ is called the interaction kernel. Thus, (185) can be written in a closed form

$$(\mathrm{i}\omega_n-\theta_\alpha\varepsilon_\alpha+\theta_\beta\varepsilon_\beta)\mathcal{G}(\alpha\beta;\gamma\delta;\omega_n)$$
$$=-S(\alpha\beta;\gamma\delta)+\sum_{\mu\nu}K(\alpha\beta;\mu\nu;\omega_n)\mathcal{G}(\mu\nu;\gamma\delta;\omega_n)\,. \quad (187)$$

Now, let us turn to the equation satisfied by $q\overline{q}$ bound states. This equation can be derived from (187) with the aid of the following Lehmann representation of the four-point Green function, which may be derived by expanding the time-ordered product in (152) and then inserting the complete set of $q\overline{q}$ bound states into (152) [15–17, 31],

$$\mathcal{G}(\alpha\beta;\gamma\delta;\omega_l)$$
$$=\frac{1}{2}\mathrm{e}^{\beta\Omega}\sum_{mn}\Delta_{mn}\left\{\frac{\chi_{nm}(\alpha\beta)\chi_{mn}(\gamma\delta)}{\mathrm{i}\omega_l-E_{nm}}-\frac{\chi_{nm}(\gamma\delta)\chi_{mn}(\alpha\beta)}{\mathrm{i}\omega_l+E_{nm}}\right\}, \quad (188)$$

where

$$\chi_{nm}(\alpha\beta)=\left\langle m\left|N\left[\widehat{b}_\alpha\widehat{b}_\beta^+\right]\right|n\right\rangle\,, \quad (189)$$

which is the transition amplitude from the state with energy E_n to the state with energy E_m, and where

$$\Delta_{nm}=\mathrm{e}^{-\beta E_n}-\mathrm{e}^{-\beta E_m}\,. \quad (190)$$

Upon substituting (188) into (187) and then taking the limit: $\lim_{\mathrm{i}\omega_l\to E_{nm}}(\mathrm{i}\omega_l-E_{nm})$, we get the following equation satisfied by the transition amplitude:

$$(E_{nm}-\theta_\alpha\varepsilon_\alpha+\theta_\beta\varepsilon_\beta)\chi_{nm}(\alpha\beta)$$
$$=\sum_{\gamma\delta}K(\alpha\beta;\gamma\delta;E_{nm})\chi_{nm}(\gamma\delta)\,, \quad (191)$$

where the fact that the function $S(\alpha\beta;\gamma\delta)$ has no bound state poles has been considered. If we take $\mid m\rangle$ to be the vacuum state $\mid 0\rangle$ and set $E=E_{n0}$ and $\chi_n(\alpha\beta)=\left\langle 0\left|N[\widehat{b}_\alpha\widehat{b}_\beta^+]\right|n\right\rangle$, we can write from the above equation that

$$(E-\theta_\alpha\varepsilon_\alpha+\theta_\beta\varepsilon_\beta)\chi_n(\alpha\beta)=\sum_{\gamma\delta}K(\alpha\beta;\gamma\delta;E)\chi_n(\gamma\delta)\,, \quad (192)$$

where the subscript n in E_n has been suppressed. This just is the equation satisfied by the $q\overline{q}$ bound states at finite temperature.

Since the index α contains $\theta_\alpha=\pm$, (192) actually is a set of coupled equations for the amplitudes $\chi_n(\alpha^+\beta^-)$, $\chi_n(\alpha^-\beta^+)$, $\chi_n(\alpha^+\beta^+)$ and $\chi_n(\alpha^-\beta^-)$. Following the procedure described in [16] and [17], one may reduce the above equation to an equivalent equation satisfied by the amplitude of positive energy. We do not repeat the derivation here. We only show the result, as follows:

$$[E-\varepsilon(\mathbf{k}_\alpha)-\varepsilon(\mathbf{k}_\beta)]\psi(\alpha\beta;E)=\sum_{\gamma\delta}V(\alpha\beta;\gamma\delta;E)\psi(\gamma\delta;E)\,, \quad (193)$$

where $\psi(\alpha\beta;E)=\chi_n(\alpha^+\beta^-)$ and $V(\alpha\beta;\gamma\delta;E)$ is the interaction Hamiltonian which can be expressed as

$$V(\alpha\beta;\gamma\delta;E)=\sum_{n=0}V^{(n)}(\alpha\beta;\gamma\delta;E)\,, \quad (194)$$

in which

$$V^{(0)}(\alpha\beta;\gamma\delta;E)=K_{++++}(\alpha\beta;\gamma\delta;E), \quad (195)$$

$$V^{(1)}(\alpha\beta;\gamma\delta;E)$$
$$=\sum_{ab\neq++}\sum_{\rho\sigma}\frac{K_{++ab}(\alpha\beta;\rho\sigma;E)K_{ab++}(\rho\sigma;\gamma\delta;E)}{E-a\varepsilon(\mathbf{k}_\rho)-b\varepsilon(\mathbf{k}_\sigma)}\,, \quad (196)$$

$$V^{(2)}(\alpha\beta;\gamma\delta;E)=\sum_{ab\neq++}\sum_{cd\neq++}\sum_{\rho\sigma}\sum_{\mu\nu}$$
$$\times\frac{K_{++ab}(\alpha\beta;\rho\sigma;E)K_{abcd}(\rho\sigma;\mu\nu;E)K_{cd++}(\mu\nu;\gamma\delta;E)}{(E-a\varepsilon(\mathbf{k}_\rho)-b\varepsilon(\mathbf{k}_\sigma))(E-c\varepsilon(\mathbf{k}_\mu)-d\varepsilon(\mathbf{k}_\nu))}\,;$$
$$\cdots\cdots. \quad (197)$$

here $a,b=\pm$, and

$$K_{++++}(\alpha\beta;\gamma\delta;E)=K(\alpha^+\beta^-;\gamma^+\delta^-;E)\,,$$
$$K_{----}(\alpha\beta;\gamma\delta;E)=K(\alpha^-\beta^+;\gamma^-\delta^+;E)\,,$$
$$K_{+-+-}(\alpha\beta;\gamma\delta;E)=K(\alpha^+\beta^+;\gamma^+\delta^+;E)\,,$$
$$K_{-+-+}(\alpha\beta;\gamma\delta;E)=K(\alpha^-\beta^-;\gamma^-\delta^-;E)\,. \quad (198)$$

6 Closed expression of the interaction kernel in the equation for $q\overline{q}$ bound states

This section is devoted to deriving a closed expression for the interaction kernel appearing in (192) and defined in (186). For this derivation, we need equations of motion which describe evolution of the Green functions $\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)$ and $\mathcal{G}(\alpha\beta\sigma;\gamma\delta;\tau_1-\tau_2)$ with time τ_2. Taking the derivatives of the generating functional in (157) with respect to $b^*_\gamma(\tau_2)$ and $b_\delta(\tau_2)$ respectively, by the same procedure as described in the derivation of (160), one may obtain

$$\left\{\frac{\mathrm{d}}{\mathrm{d}\tau_2}\frac{\delta}{\delta\eta^*_\gamma(\tau_2)}+\theta_\gamma\varepsilon_\gamma\frac{\delta}{\delta\eta^*_\gamma(\tau_2)} +\sum_{\rho\lambda}\theta_\gamma\theta_\rho A(\gamma\rho\lambda)\frac{\delta^2}{\delta\eta^*_\rho(\tau_2)\delta j^*_\lambda(\tau_2)}-\eta_\gamma(\tau_2)\right\}Z[j]=0\,. \tag{199}$$

and

$$\left\{\frac{d}{\mathrm{d}\tau_2}\frac{\delta}{\delta\eta_\delta(\tau_2)}-\theta_\delta\varepsilon_\delta\frac{\delta}{\delta\eta_\delta(\tau_2)} -\sum_{\sigma\lambda}\theta_\delta\theta_\sigma A(\sigma\delta\lambda)\frac{\delta^2}{\delta\eta_\sigma(\tau_2)\delta j^*_\lambda(\tau_2)}-\eta^*_\delta(\tau_2)\right\}Z[j]=0\,. \tag{200}$$

Performing differentiations of (199) and (200) with respect to the sources $\eta_\delta(\tau_2)$ and $\eta^*_\gamma(\tau_2)$ respectively, we get

$$\left\{\left(\frac{\mathrm{d}}{\mathrm{d}\tau_2}\frac{\delta}{\delta\eta^*_\gamma(\tau_2)}\right)\frac{\delta}{\delta\eta_\delta(\tau_2)}+\theta_\gamma\varepsilon_\gamma\frac{\delta^2}{\delta\eta^*_\gamma(\tau_2)\delta\eta_\delta(\tau_2)} +\sum_{\rho\lambda}\theta_\gamma\theta_\rho A(\gamma\rho\lambda)\frac{\delta^3}{\delta\eta^*_\rho(\tau_2)\delta\eta_\delta(\tau_2)\delta j^*_\lambda(\tau_2)} +\delta_{\gamma\delta}-\eta_\gamma(\tau_2)\frac{\delta}{\delta\eta_\delta(\tau_2)}\right\}Z[j]=0 \tag{201}$$

and

$$\left\{\frac{\delta}{\delta\eta^*_\gamma(\tau_2)}\left(\frac{\mathrm{d}}{\mathrm{d}\tau_2}\frac{\delta}{\delta\eta_\delta(\tau_2)}\right)-\theta_\delta\varepsilon_\delta\frac{\delta}{\delta\eta^*_\gamma(\tau_2)\delta\eta_\delta(\tau_2)} -\sum_{\sigma\lambda}\theta_\delta\theta_\sigma A(\sigma\delta\lambda)\frac{\delta^3}{\delta\eta^*_\gamma(\tau_2)\delta\eta_\sigma(\tau_2)\delta j^*_\lambda(\tau_2)} -\delta_{\gamma\delta}+\eta^*_\delta(\tau_2)\frac{\delta}{\delta\eta^*_\gamma(\tau_2)}\right\}Z[j]=0. \tag{202}$$

Furthermore, by successively differentiating (201) and (202) with respect to the sources $\eta^*_\alpha(\tau_1)$ and $\eta_\beta(\tau_1)$, one obtains

$$\left\{\frac{\delta^2}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\beta(\tau_1)}\left(\frac{\mathrm{d}}{\mathrm{d}\tau_2}\frac{\delta}{\delta\eta^*_\gamma(\tau_2)}\right)\frac{\delta}{\delta\eta_\delta(\tau_2)} +\theta_\gamma\varepsilon_\gamma\frac{\delta^4}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta^*_\gamma(\tau_2)\delta\eta_\delta(\tau_2)} +\sum_{\lambda\sigma}\theta_\gamma\theta_\lambda A(\gamma\rho\lambda)\frac{\delta^5}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta^*_\rho(\tau_2)\delta\eta_\delta(\tau_2)\delta j^*_\lambda(\tau_2)} +\delta_{\gamma\delta}\frac{\delta^2}{\delta\eta^*_\alpha(\tau_2)\delta\eta_\beta(\tau_2)}-\delta_{\beta\gamma}\delta(\tau_1-\tau_2)\frac{\delta^2}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\delta(\tau_2)} -\eta_\gamma(\tau_2)\frac{\delta^3}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta_\delta(\tau_2)}\right\}Z[j]=0 \tag{203}$$

and

$$\left\{\frac{\delta^3}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta^*_\gamma(\tau_2)}\left(\frac{\mathrm{d}}{\mathrm{d}\tau_2}\frac{\delta}{\delta\eta_\delta(\tau_2)}\right) -\theta_\delta\varepsilon_\delta\frac{\delta^4}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta^*_\gamma(\tau_2)\delta\eta_\delta(\tau_2)} -\sum_{\sigma\lambda}\theta_\delta\theta_\sigma A(\sigma\delta\lambda)\frac{\delta^5}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta^*_\gamma(\tau_2)\delta\eta_\sigma(\tau_2)\delta j^*_\lambda(\tau_2)} -\delta_{\gamma\delta}\frac{\delta^2}{\delta\eta^*_\alpha(\tau_2)\delta\eta_\beta(\tau_2)}+\delta_{\alpha\delta}\delta(\tau_1-\tau_2)\frac{\delta^2}{\delta\eta^*_\gamma(\tau_2)\delta\eta_\beta(\tau_1)} +\eta^*_\delta(\tau_2)\frac{\delta^3}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta^*_\gamma(\tau_2)}\right\}Z[j]=0. \tag{204}$$

Let us sum up (201) and (202) at first, then multiply both sides of the equation thus obtained with $-\theta_\gamma\theta_\delta$ and finally set all the sources but the source j_λ to vanish. By these operations, we get

$$\sum_{\rho\sigma}\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}S^{j_\lambda}_{\rho\sigma}=0\,, \tag{205}$$

where

$$\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}=\left(\frac{\mathrm{d}}{\mathrm{d}\tau_2}+\theta_\gamma\varepsilon_\gamma-\theta_\delta\varepsilon_\delta\right)\delta_{\gamma\rho}\delta_{\delta\sigma} -\sum_\lambda f(\rho\sigma\lambda;\gamma\delta)\frac{\delta}{\delta j^*_\lambda(\tau_2)} \tag{206}$$

in which

$$f(\rho\sigma\lambda;\gamma\delta)=A(\sigma\delta\lambda)\delta_{\gamma\rho}-A(\gamma\rho\lambda))\delta_{\delta\sigma}=-f(\gamma\delta;\rho\sigma\lambda) \tag{207}$$

and $S^{j_\lambda}_{\rho\sigma}$ was defined in (167).

When we sum up (203) and (204), then multiply both sides of the equation thus obtained with $\theta_\alpha\theta_\beta\theta_\gamma\theta_\delta$ and finally set all the sources but the source j_λ to be zero, according to the definitions in (172) and (173), it is found that

$$\sum_{\rho\sigma}\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}G(\alpha\beta;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}$$
$$=\delta(\tau_1-\tau_2)\left[\delta_{\alpha\delta}S_{\gamma\beta}(\tau_2-\tau_1)^{j_\lambda}-\delta_{\beta\gamma}S_{\alpha\delta}(\tau_1-\tau_2)^{j_\lambda}\right]. \quad (208)$$

In order to derive the equation of motion satisfied by the Green function $G(\lambda\tau\sigma;\gamma\delta;\tau_1-\tau_2)$ defined in (183), we may take the derivative of (208) with respect to $j_\lambda^*(\tau_1)$. In this way, we get

$$\sum_{\rho\sigma}\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}G(\alpha\beta\lambda;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}$$
$$=\delta(\tau_1-\tau_2)\left[\delta_{\alpha\delta}\Lambda(\gamma\beta\rho;\tau_2-\tau_1)^{j_\lambda}-\delta_{\beta\gamma}\Lambda(\alpha\delta\rho;\tau_1-\tau_2)^{j_\lambda}\right], \quad (209)$$

where

$$\Lambda(\gamma\beta\rho;\tau_2-\tau_1)^{j_\lambda}=\frac{\delta}{\delta j_\lambda^*(\tau_1)}S_{\gamma\beta}(\tau_2-\tau_1)^{j_\lambda}, \quad (210)$$

$$\Lambda(\alpha\delta\rho;\tau_1-\tau_2)^{j_\lambda}=\frac{\delta}{\delta j_\lambda^*(\tau_1)}S_{\alpha\delta}(\tau_1-\tau_2)^{j_\lambda} \quad (211)$$

and

$$G(\alpha\beta\lambda;\gamma\delta;\tau_1-\tau_2)=\frac{\delta}{\delta j_\lambda^*(\tau_1)}G(\alpha\beta;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}. \quad (212)$$

Acting on (155) and (182) with the operator $\overline{H}(\gamma\delta;\rho\sigma;\tau_2)$ and employing (205), we find

$$\sum_{\rho\sigma}\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}\mathcal{G}(\alpha\beta;\rho\sigma;\tau_1-\tau_2)^{j_\lambda}$$
$$=\sum_{\rho\sigma}\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}G(\alpha\beta;\rho\sigma;\tau_1-\tau_2)^{j_\lambda} \quad (213)$$

and

$$\sum_{\rho\sigma}\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}\mathcal{G}(\alpha\beta\lambda;\rho\sigma;\tau_1-\tau_2)^{j_\lambda}$$
$$=\sum_{\rho\sigma}\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}G(\alpha\beta\lambda;\rho\sigma;\tau_1-\tau_2)^{j_\lambda}. \quad (214)$$

The above two equalities further indicate that the equations of motion satisfied by the Green functions $\mathcal{G}(\alpha\beta;\rho\sigma;\tau_1-\tau_2)^{j_\lambda}$ and $\mathcal{G}(\alpha\beta\lambda;\rho\sigma;\tau_1-\tau_2)^{j_\lambda}$ are formally the same as those for the ordinary Green functions $G(\alpha\beta;\rho\sigma;\tau_1-\tau_2)^{j_\lambda}$ and $G(\alpha\beta\lambda;\rho\sigma;\tau_1-\tau_2)^{j_\lambda}$ respectively. Upon inserting (208) into (213) and (209) into (214) and turning off the source j_λ, noticing the definition in (206), we derive the following equations:

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau_2}+\theta_\gamma\varepsilon_\gamma-\theta_\delta\varepsilon_\delta\right)\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)$$
$$=\delta(\tau_1-\tau_2)[\delta_{\alpha\delta}S_{\gamma\beta}(\tau_2-\tau_1)-\delta_{\beta\gamma}S_{\alpha\delta}(\tau_1-\tau_2)]$$
$$+\sum_{\lambda\tau\sigma}\mathcal{G}(\alpha\beta;\lambda\tau\sigma;\tau_1-\tau_2)f(\lambda\tau\sigma;\gamma\delta) \quad (215)$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau_2}+\theta_\gamma\varepsilon_\gamma-\theta_\delta\varepsilon_\delta\right)\mathcal{G}(\alpha\beta\rho;\gamma\delta;\tau_1-\tau_2)$$
$$=\delta(\tau_1-\tau_2)\left[\delta_{\alpha\delta}\Lambda(\gamma\beta\rho;\tau_2-\tau_1)-\delta_{\beta\gamma}\Lambda(\alpha\delta\rho;\tau_1-\tau_2)\right]$$
$$+\sum_{\lambda\tau\sigma}\mathcal{G}(\alpha\beta\rho;\lambda\tau\sigma;\tau_1-\tau_2)f(\lambda\tau\sigma;\gamma\delta), \quad (216)$$

where some indices have been changed for convenience:

$$\Lambda(\gamma\beta\rho;\tau_2-\tau_1)=\left\langle T\left[\widehat{b}_\gamma(\tau_2)\widehat{b}_\beta^+(\tau_1)\widehat{a}_\rho(\tau_1)\right]\right\rangle_\beta,$$
$$\Lambda(\alpha\delta\rho;\tau_1-\tau_2)=\left\langle T\left[\widehat{b}_\alpha(\tau_1)\widehat{b}_\delta^+(\tau_2)\widehat{a}_\rho(\tau_1)\right]\right\rangle_\beta, \quad (217)$$

which are given by (210) and (211) on setting $j_\lambda=0$, and

$$\mathcal{G}(\lambda\tau\rho;\gamma\delta\sigma;\tau_1-\tau_2)$$
$$=\frac{\delta^2}{\delta j_\rho^*(\tau_1)\delta j_\sigma^*(\tau_2)}\mathcal{G}(\lambda\tau;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}\Big|_{j_\lambda=0}$$
$$=\left\langle T\left\{N\left[\widehat{b}_\lambda(\tau_1)\widehat{b}_\tau^+(\tau_1)\widehat{a}_\rho(\tau_1)\right]N\left[\widehat{b}_\gamma(\tau_2)\widehat{b}_\delta^+(\tau_2)\widehat{a}_\sigma(\tau_2)\right]\right\}\right\rangle_\beta \quad (218)$$

is the six-point Green function including two gluon operators in it. According to the definition in (180), we have

$$\mathcal{G}(\lambda\tau\rho;\gamma\delta\sigma;\tau_1-\tau_2)=G(\lambda\tau\rho;\gamma\delta\sigma;\tau_1-\tau_2)-\Lambda(\lambda\tau\rho)\Lambda(\gamma\delta\sigma), \quad (219)$$

where

$$G(\lambda\tau\rho;\gamma\delta\sigma;\tau_1-\tau_2)$$
$$=\left\langle T\left[\widehat{b}_\lambda(\tau_1)\widehat{b}_\tau^+(\tau_1)\widehat{a}_\rho(\tau_1)\widehat{b}_\gamma(\tau_2)\widehat{b}_\delta^+(\tau_2)\widehat{a}_\sigma(\tau_2)\right]\right\rangle_\beta \quad (220)$$

is the ordinary six-point Green function. It should be noted that due to the restriction of the delta function, the terms in the brackets on the right hand sides of (215) and (216) actually are "time"-independent.

It is easy to see that the Green functions $\mathcal{G}(\alpha\beta;\lambda\tau\sigma;\tau_1-\tau_2)$ and $\mathcal{G}(\lambda\tau\rho;\gamma\delta\sigma;\tau_1-\tau_2)$, as the Green functions $\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)$ and $\mathcal{G}(\alpha\beta\rho;\gamma\delta;\tau_1-\tau_2)$, are periodic. Therefore, by the Fourier transformation, i.e. by the integration $\int_0^\beta \mathrm{d}\tau \mathrm{e}^{\mathrm{i}\omega_n\tau}$, noticing $\mathrm{d}/\mathrm{d}\tau_2=-\mathrm{d}/\mathrm{d}\tau$, (215) and (216) will be transformed to

$$(\mathrm{i}\omega_n+\theta_\gamma\varepsilon_\gamma-\theta_\delta\varepsilon_\delta)\mathcal{G}(\alpha\beta;\gamma\delta;\omega_n)$$
$$=S(\alpha\beta;\gamma\delta)+\sum_{\lambda\tau\sigma}\mathcal{G}(\alpha\beta;\lambda\tau\sigma;\omega_n)f(\lambda\tau\sigma;\gamma\delta), \quad (221)$$

where $S(\alpha\beta;\gamma\delta)$ was defined in (179) and

$$(\mathrm{i}\omega_n+\theta_\gamma\varepsilon_\gamma-\theta_\delta\varepsilon_\delta)\mathcal{G}(\alpha\beta\rho;\gamma\delta;\omega_n)$$
$$=R(\alpha\beta\rho;\gamma\delta)+\sum_{\lambda\tau\sigma}\mathcal{G}(\alpha\beta\rho;\lambda\tau\sigma;\omega_n)f(\lambda\tau\sigma;\gamma\delta), \quad (222)$$

where

$$R(\alpha\beta\rho;\gamma\delta) = \delta_{\alpha\delta}\Lambda(\gamma\beta\rho) - \delta_{\beta\gamma}\Lambda(\alpha\delta\rho) = \left\langle \left[\widehat{b}_\alpha \widehat{b}^+_\beta, \widehat{b}_\gamma \widehat{b}^+_\delta\right]_- \widehat{a}_\rho \right\rangle_\beta , \quad (223)$$

which is "time"-independent.

Now we are ready to derive the interaction kernel. Acting on both sides of (186) with $(\mathrm{i}\omega_n + \theta_\gamma\varepsilon_\gamma - \theta_\delta\varepsilon_\delta)$ and using (221) and (222), one gets

$$\begin{aligned}\sum_{\mu\nu} K(\alpha\beta;\mu\nu;\omega_n)S(\mu\nu;\gamma\delta) &= \sum_{\lambda\tau\rho} f(\alpha\beta;\lambda\tau\rho)R(\lambda\tau\rho;\gamma\delta) \\ &+ \sum_{\lambda\tau\rho}\sum_{\xi\eta\sigma} f(\alpha\beta;\lambda\tau\rho)\mathcal{G}(\lambda\tau\rho;\xi\eta\sigma;\omega_n)f(\xi\eta\sigma;\gamma\delta) \\ &- \sum_{\mu\nu}\sum_{\xi\eta\sigma} K(\alpha\beta;\mu\nu;\omega_n)\mathcal{G}(\alpha\beta;\xi\eta\sigma;\omega_n)f(\lambda\tau\sigma;\gamma\delta) .\end{aligned} \quad (224)$$

Operating on both sides of (186) with the inverse of $\mathcal{G}(\mu\nu;\gamma\delta;\omega_n)$, we have

$$K(\alpha\beta;\gamma\delta;\omega_n) = \sum_{\gamma\delta}\sum_{\lambda\tau\sigma} f(\alpha\beta;\lambda\tau\rho)\mathcal{G}(\lambda\tau\rho;\mu\nu;\omega_n)\mathcal{G}^{-1}(\mu\nu;\gamma\delta;\omega_n) . \quad (225)$$

Upon substituting (225) onto the right hand side of (224) and acting on (224) with the inverse $S^{-1}(\mu\nu;\gamma\delta)$, we eventually arrive at

$$\begin{aligned}K(\alpha\beta;\gamma\delta;E) &= \sum_{\mu\nu}\Bigg\{\sum_{\lambda\tau\rho} f(\alpha\beta;\lambda\tau\rho)R(\lambda\tau\rho;\mu\nu) \\ &+ \sum_{\lambda\tau\rho}\sum_{\xi\eta\sigma} f(\alpha\beta;\lambda\tau\rho)\mathcal{G}(\lambda\tau\rho;\xi\eta\sigma;E)f(\xi\eta\sigma;\mu\nu) \\ &- \sum_{\lambda\tau\rho}\sum_{\xi\eta\sigma}\sum_{\kappa\varsigma}\sum_{\pi\theta} f(\alpha\beta;\lambda\tau\rho)\mathcal{G}(\lambda\tau\rho;\kappa\varsigma;E) \\ &\times \mathcal{G}^{-1}(\kappa\varsigma;\pi\theta;E)\mathcal{G}(\pi\theta;\xi\eta\sigma;E)f(\xi\eta\sigma;\mu\nu)\Bigg\} \\ &\times S^{-1}(\mu\nu;\gamma\delta) ,\end{aligned} \quad (226)$$

where ω_n has been replaced by E. This just is the wanted closed expression of the interaction kernel appearing in (192). In accordance with (186), the last term in (226) can be written in the form

$$\sum_{\rho\sigma}\sum_{\xi\eta}\sum_{\mu\nu} K(\alpha\beta;\rho\sigma;E)\mathcal{G}(\rho\sigma;\xi\eta;E) \times K(\xi\eta;\mu\nu;E)S^{-1}(\mu\nu;\gamma\delta) \quad (227)$$

which exhibits a typical B-S reducible structure [17]. Therefore, the last term in (226) plays the role of cancelling the B-S reducible part contained in the other terms in (226) to make the kernel to be B-S irreducible. If we use the above expression in place of the last term in (226) and acting on (226) with $S(\gamma\delta;\mu\nu)$, we obtain from (226) an integral equation satisfied by the kernel $K(\alpha\beta;\gamma\delta;E)$. Define

$$\mathcal{R}(\alpha\beta;\gamma\delta) = \sum_{\lambda\tau\rho} f(\alpha\beta;\lambda\tau\rho)R(\lambda\tau\rho;\gamma\delta) \quad (228)$$

and

$$\mathcal{Q}(\alpha\beta;\gamma\delta) = \sum_{\lambda\tau\rho}\sum_{\xi\eta\sigma} f(\alpha\beta;\lambda\tau\rho)\mathcal{G}(\lambda\tau\rho;\xi\eta\sigma;E)f(\xi\eta\sigma;\gamma\delta), \quad (229)$$

the integral equation can be written in the matrix form as follows:

$$KS = \mathcal{R} + \mathcal{Q} - K\mathcal{G}K. \quad (230)$$

For comparison with the kernel in (226) and for convenience of nonperturbative investigations, we would like to show the corresponding closed expression given in the position space without giving derivation. This kernel can be obtained from the kernel in (226) by making use of the inverse of the Fourier transformations written in Sect. 3 or derived from the generating functional represented in the position space (see appendix) by completely following the procedure as described in this section. The kernel is represented as follows:

$$\begin{aligned}K(\mathbf{x}_1,\mathbf{x}_2;\mathbf{y}_1,\mathbf{y}_2;E) &= \int \mathrm{d}^3z_1\mathrm{d}^3z_2\{\mathcal{R}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{z}_1,\mathbf{z}_2) + \mathcal{Q}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{z}_1,\mathbf{z}_2;E) \\ &- \mathcal{D}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{z}_1,\mathbf{z}_2;E)\}S^{-1}(\mathbf{z}_1,\mathbf{z}_2;\mathbf{y}_1,\mathbf{y}_2) ;\end{aligned} \quad (231)$$

$\mathcal{R}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{z}_1,\mathbf{z}_2)$, $\mathcal{Q}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{z},\mathbf{z}_2;E)$ and $\mathcal{D}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{z}_1,\mathbf{z}_2;E)$ will now be described:

$$\mathcal{R}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{z}_1,\mathbf{z}_2) = \sum_{i=1}^{2} \Omega_i^{a\mu}\mathcal{R}_\mu^{(i)a}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{z}_1,\mathbf{z}_2) \quad (232)$$

in which

$$\Omega_1^{a\mu} = \mathrm{i}g\gamma_1^4\gamma_1^\mu T_1^a,\ \Omega_2^{b\nu} = \mathrm{i}g\gamma_2^4\gamma_2^\nu \overline{T}_2^b , \quad (233)$$

with $T_1^a = \lambda^a/2$ and $\overline{T}_2^b = -\lambda^{a*}/2$ being the quark and antiquark color matrices respectively and

$$\begin{aligned}\mathcal{R}_\mu^{(i)a}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{z}_1,\mathbf{z}_2) &= \delta^3(\mathbf{x}_1-\mathbf{z}_1)\gamma_1^4\Lambda_\mu^{\mathbf{c}a}(\mathbf{x}_i \mid \mathbf{x}_2,\mathbf{z}_2) \\ &+ \delta^3(\mathbf{x}_2-\mathbf{z}_2)\gamma_2^4\Lambda_\mu^a(\mathbf{x}_i \mid \mathbf{x}_1,\mathbf{z}_1) ;\end{aligned} \quad (234)$$

here $\Lambda_\mu^a(\mathbf{x}_i \mid \mathbf{x}_1,\mathbf{y}_1)$ and $\Lambda_\mu^{\mathbf{c}a}(\mathbf{x}_i \mid \mathbf{x}_2,\mathbf{y}_2)$ are defined as

$$\begin{aligned}\Lambda_\mu^a(\mathbf{x}_i \mid \mathbf{x}_1,\mathbf{y}_1) &= \left\langle T\left[\mathbf{A}_\mu^a(\mathbf{x}_i,\tau_1)\boldsymbol{\psi}(\mathbf{x}_1,\tau_1)\overline{\boldsymbol{\psi}}(\mathbf{y}_1,\tau_1)\right]\right\rangle_\beta , \\ \Lambda_\mu^{\mathbf{c}a}(\mathbf{x}_i \mid \mathbf{x}_2,\mathbf{y}_2) &= \left\langle T\left[\mathbf{A}_\mu^a(\mathbf{x}_i,\tau_1)\boldsymbol{\psi}^c(\mathbf{x}_2,\tau_1)\overline{\boldsymbol{\psi}}^c(\mathbf{y}_2,\tau_1)\right]\right\rangle_\beta ,\end{aligned} \quad (235)$$

which are time-independent due to the translation-invariance property of the Green functions. Also, we have

$$\mathcal{Q}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{z}_1,\mathbf{z}_2;E) = \sum_{i,j=1}^{2} \Omega_i^{a\mu}\mathcal{G}_{\mu\nu}^{ab}(\mathbf{x}_i,\mathbf{z}_j \mid \mathbf{x}_1,\mathbf{x}_2;\mathbf{z}_1,\mathbf{z}_2;E)\overline{\Omega}_j^{b\nu}, \tag{236}$$

in which

$$\overline{\Omega}_1^{a\mu} = \mathrm{i}g\gamma_1^\mu\gamma_1^4 T_1^a,\ \overline{\Omega}_2^{a\mu} = \mathrm{i}g\gamma_2^\mu\gamma_2^4\overline{T}_2^a, \tag{237}$$

and $\mathcal{G}_{\mu\nu}^{ab}(\mathbf{x}_i,\mathbf{z}_j \mid \mathbf{x}_1,\mathbf{x}_2;\mathbf{z}_1,\mathbf{z}_2;E)$ is the Fourier transform of the Green function defined by

$$\begin{aligned}&\mathcal{G}_{\mu\nu}^{ab}(\mathbf{x}_i,\mathbf{z}_j \mid \mathbf{x}_1,\mathbf{x}_2;\mathbf{z}_1,\mathbf{z}_2;\tau_1-\tau_2)\\ &= \Big\langle T\Big\{N\left[\mathbf{A}_\mu^a(\mathbf{x}_i,\tau_1)\boldsymbol{\psi}(\mathbf{x}_1,\tau_1)\boldsymbol{\psi}^c(\mathbf{x}_2,\tau_1)\right]\\ &\quad\times N\left[\mathbf{A}_\nu^b(\mathbf{z}_j,\tau_2)\overline{\boldsymbol{\psi}}(\mathbf{z}_1,\tau_2)\overline{\boldsymbol{\psi}}^c(\mathbf{z}_2,\tau_2)\right]\Big\}\Big\rangle_\beta\end{aligned} \tag{238}$$

and

$$\begin{aligned}&\mathcal{D}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{z}_1,\mathbf{z}_2;E)\\ &= \int\prod_{k=1}^{2}\mathrm{d}^3u_k\mathrm{d}^3v_k\sum_{i,j=1}^{2}\Omega_i^{a\mu}\mathcal{G}_\mu^{(i)a}(\mathbf{x}_i \mid \mathbf{x}_1,\mathbf{x}_2;\mathbf{u}_1,\mathbf{u}_2;E)\\ &\quad\times\mathcal{G}^{-1}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{v}_1,\mathbf{v}_2;E)\mathcal{G}_\nu^{(j)b}(\mathbf{z}_j \mid \mathbf{v}_1,\mathbf{v}_2;\mathbf{z}_1,\mathbf{z}_2;E)\overline{\Omega}_j^{b\nu};\end{aligned} \tag{239}$$

in which $\mathcal{G}_\mu^{(i)a}(\mathbf{x}_i \mid \mathbf{x}_1,\mathbf{x}_2;\mathbf{u}_1,\mathbf{u}_2;E)$, and $\mathcal{G}_\nu^{(j)b}(\mathbf{z}_j \mid \mathbf{v}_1,\mathbf{v}_2;\mathbf{z}_1,\mathbf{z}_2;E)$ are the Fourier transforms of the following Green functions:

$$\begin{aligned}&\mathcal{G}_\mu^{(i)a}(\mathbf{x}_i \mid \mathbf{x}_1,\mathbf{x}_2;\mathbf{u}_1,\mathbf{u}_2;\tau_1-\tau_2)\\ &= \Big\langle T\Big\{N\left[\mathbf{A}_\mu^a(\mathbf{x}_i,\tau_1)\boldsymbol{\psi}(\mathbf{x}_1,\tau_1)\boldsymbol{\psi}^c(\mathbf{x}_2,\tau_1)\right]\\ &\quad\times N\left[\overline{\boldsymbol{\psi}}(\mathbf{u}_1,\tau_2)\overline{\boldsymbol{\psi}}^c(\mathbf{u}_2,\tau_2)\right]\Big\}\Big\rangle_\beta\end{aligned} \tag{240}$$

and

$$\begin{aligned}&\mathcal{G}_\nu^{(j)b}(\mathbf{z}_j \mid \mathbf{v}_1,\mathbf{v}_2;\mathbf{z}_1,\mathbf{z}_2;\tau_1-\tau_2)\\ &= \Big\langle T\Big\{N\left[\boldsymbol{\psi}(\mathbf{v}_1,\tau_1)\boldsymbol{\psi}^c(\mathbf{v}_2,\tau_1)\right]\\ &\quad\times N\left[\mathbf{A}_\nu^b(\mathbf{z}_j,\tau_2)\overline{\boldsymbol{\psi}}(\mathbf{z}_1,\tau_2)\overline{\boldsymbol{\psi}}^c(\mathbf{z}_2,\tau_2)\right]\Big\}\Big\rangle_\beta .\end{aligned} \tag{241}$$

The $S^{-1}(\mathbf{z}_1,\mathbf{z}_2;\mathbf{y}_1,\mathbf{y}_2)$ in (231) is the inverse of the function defined by

$$\begin{aligned}S(\mathbf{x}_1,\mathbf{x}_2;\mathbf{z}_1,\mathbf{z}_2) &= \delta^3(\mathbf{x}_1-\mathbf{z}_1)\gamma_1^4 S_{\mathrm{F}}^{\mathrm{c}}(\mathbf{x}_2-\mathbf{z}_2)\\ &\quad+\delta^3(\mathbf{x}_2-\mathbf{z}_2)\gamma_2^4 S_{\mathrm{F}}(\mathbf{x}_1-\mathbf{z}_1),\end{aligned} \tag{242}$$

in which $S_{\mathrm{F}}(\mathbf{x}_1-\mathbf{z}_1)$ and $S_{\mathrm{F}}^{\mathrm{c}}(\mathbf{x}_2-\mathbf{z}_2)$ are the equal-time quark and antiquark thermal propagators respectively. It is clear that there is one-to-one correspondence between both kernels written in (226) and (231). It is noted that the interaction kernel derived in this section is nonperturbative because the Green functions included in the kernel are defined in the Heisenberg picture. Perturbative calculations of the kernel can easily be done by using the familiar perturbative expansions of Green functions as illustrated in the next section.

7 One gluon exchange kernel and Hamiltonian

In this section, we would like to show the one-gluon exchange kernel given by the expression in (226). In the lowest order approximation of perturbation, only the first term of the series in (194), i.e., the kernel $K(\alpha^+\beta^-;\gamma^+\delta^-;E)$ represented in (195), is needed to be taken into account in (192) and (193). In the lowest order approximation, as seen from (223), the first term in (226) vanishes because the function $R(\lambda\tau\rho;\gamma^+\delta^-)$ gives no contribution to the kernel owing to the vanishing expectation value $\langle n\,|\widehat{a}_\rho|\,n\rangle$. Therefore, the one-gluon exchange kernel can only arise from the second term in (226) where the Green function $\mathcal{G}(\lambda\tau\rho;\xi\eta\sigma;E)$ reduces to the ordinary one $G(\lambda\tau\rho;\xi\eta\sigma;E)$ in the lowest order approximation because the last term in (226) vanishes in this case for the reason as argued for the function $R(\lambda\tau\rho;\gamma^+\delta^-)$. To evaluate the inverse $S^{-1}(\mu\nu;\gamma^+\delta^-)$, we first evaluate $S(\gamma^+\delta^-;\mu\nu)$. From the expression denoted in (179), it is easy to find that the nonvanishing contribution of $S(\gamma^+\delta^-;\mu\nu)$ is given by

$$\begin{aligned}&S(\gamma^+\delta^-;\mu^-\nu^+)\\ &= \delta_{\gamma^+\nu^+}S_{\mu^-\delta^-} - \delta_{\mu^-\delta^-}S_{\gamma^+\nu^+}\\ &= \delta_{\gamma^+\nu^+}\delta_{\mu^-\delta^-} - \delta_{\mu^-\delta^-}S_{\gamma^+\nu^+} - \delta_{\gamma^+\nu^+}\overline{S}_{\delta^-\mu^-},\end{aligned} \tag{243}$$

where $S_{\gamma^+\nu^+}$ and $\overline{S}_{\delta^-\mu^-}$ are the quark and antiquark equal-time propagators. Their expressions, according to the common definition, can be read from (139) and (140) by setting $\tau_1-\tau_2\to 0^+$:

$$\begin{aligned}S_{\gamma^+\nu^+} &= \delta_{\gamma^+\nu^+}\Delta_q(\mathbf{q}_1,0^+) = \delta_{\gamma^+\nu^+}\frac{1}{2}\left[\overline{n}_f(\mathbf{q}_1)-n_f(\mathbf{q}_1)\right],\\ \overline{S}_{\delta^-\mu^-} &= \delta_{\delta^-\mu^-}\Delta_q(\mathbf{q}_2,0^+) = \delta_{\delta^-\mu^-}\frac{1}{2}\left[\overline{n}_f(\mathbf{q}_2)-n_f(\mathbf{q}_2)\right].\end{aligned} \tag{244}$$

Therefore, we can write

$$S^{-1}(\mu^-\nu^+;\gamma^+\delta^-) = \delta_{\gamma^+\nu^+}\delta_{\mu^-\delta^-}\mathcal{S}^{-1}(\gamma\delta), \tag{245}$$

where

$$\mathcal{S}(\gamma\delta) = 1-\frac{1}{2}\left[\overline{n}_f(\mathbf{q}_1)-n_f(\mathbf{q}_1)+\overline{n}_f(\mathbf{q}_2)-n_f(\mathbf{q}_2)\right]. \tag{246}$$

Thus, to derive the lowest order approximate kernel, we only need to consider

$$K(\alpha^+\beta^-;\gamma^+\delta^-;E) = \sum_{\lambda\tau\rho}\sum_{\xi\zeta\sigma} f(\alpha^+\beta^-;\lambda\tau\rho)G(\lambda\tau\rho;\mu\nu\sigma;E) \times f(\mu\nu\sigma;\delta^-\gamma^+)\mathcal{S}^{-1}(\gamma\delta)\,. \quad (247)$$

From (169), (207), (80) and (67), it is clearly seen that when λ, τ, μ and ν take the values λ^+, τ^-, μ^- and ν^+, the functions $f(\alpha^+\beta^-;\lambda\tau\rho)$ and $f(\mu\nu\sigma;\delta^-\gamma^+)$ give the quark–antiquark interaction taking place in the t-channel scattering process; while, when λ, τ, μ and ν take the values λ^-, τ^+, μ^+ and ν^-, the $f(\alpha^+\beta^-;\lambda\tau\rho)$ and $f(\mu\nu\sigma;\delta^-\gamma^+)$ will give the quark–antiquark vertices which describe the $q\overline{q}$ annihilation process. Since the expectation value of the $q\overline{q}$ color matrix appearing in the $q\overline{q}$ lowest order annihilation process is zero in the color singlet, it is only necessary to consider the following interaction kernel:

$$\begin{aligned}K(\alpha^+\beta^-;\gamma^+\delta^-;E) =&\sum_{\mu\nu}\sum_{\lambda\tau\rho}\sum_{\xi\eta\sigma}\Big[f(\alpha^+\beta^-;\lambda^+\tau^-\rho^+)G(\lambda^+\tau^-\rho^+;\mu^-\nu^+\sigma^-;E)\\ &\times f(\mu^-\nu^+\sigma^-;\delta^-\gamma^+)+f(\alpha^+\beta^-;\lambda^+\tau^-\rho^-)\\ &\times G(\lambda^+\tau^-\rho^-;\mu^-\nu^+\sigma^+;E)f(\mu^-\nu^+\sigma^+;\delta^-\gamma^+)\Big]\mathcal{S}^{-1}(\gamma\delta).\end{aligned} \quad (248)$$

Noting that the functions $f(\alpha^+\beta^-;\lambda^+\tau^-\rho)$ and $f(\mu^-\nu^+\sigma;\delta^-\gamma^+)$ are proportional to the coupling constant g, in the approximation of order g^2, we only need to consider the zero-order of the Green function $G(\lambda^+\tau^-\rho^\pm;\mu^-\nu^+\sigma^\mp;E)$. This Green function may easily be derived from the generating functional $Z^0[j]$ represented in (96), (119), (138) and (150). The result is

$$\begin{aligned}&G(\lambda^+\tau^-\rho^-;\mu^-\nu^+\sigma^+;\tau_1-\tau_2)\\ &=\frac{1}{Z^0}\frac{\delta^6 Z^0[j\}}{\delta\eta^*_\lambda(\tau_1)\delta\overline{\eta}^*_\tau(\tau_1)\delta\overline{\eta}_\mu(\tau_2)\delta\eta_\nu(\tau_2)\delta\xi^*_\rho(\tau_1)\delta\xi_\sigma(\tau_2)}\Big|_{j=0}\\ &=\delta_{\lambda\nu}\Delta_q(\mathbf{k}_\lambda,\tau_1-\tau_2)\delta_{\tau\mu}\Delta_q(\mathbf{k}_\tau,\tau_1-\tau_2)\delta_{\rho\sigma}\Delta_g(\mathbf{k}_\rho,\tau_1-\tau_2)\\ &=G(\lambda^+\tau^-\rho^+;\mu^-\nu^+\sigma^-;\tau_1-\tau_2)\,.\end{aligned} \quad (249)$$

The last equality in the above arises from the fact that $\Delta_g(\mathbf{k}_\rho,\tau_1-\tau_2)$ is an even function of $\tau_1-\tau_2$. With the expressions given in (124) and (140), the Fourier transform of $G(\lambda^+\tau^-\rho^-;\mu^-\nu^+\sigma^+;\tau_1-\tau_2)$ can be found to be

$$G(\lambda^+\tau^-\rho^-;\mu^-\nu^+\sigma^+;\omega_n)=\delta_{\lambda\nu}\delta_{\tau\mu}\delta_{\rho\sigma}\frac{1}{8}\Delta(\lambda\tau\rho)\,, \quad (250)$$

where $\Delta(\lambda\tau\rho)$ will be specified soon.

Substitution of (169), (207) and (250) in (248) leads to

$$\begin{aligned}&K(\alpha^+\beta^-;\gamma^+\delta^-;E)\\ &=\frac{1}{4}\Big[\sum_{\lambda\rho}A(\alpha^+\lambda^+\rho^+)A(\lambda^+\gamma^+\rho^-)\Delta(\lambda\beta\rho)\delta_{\beta\delta}\\ &\quad+\sum_{\tau\rho}A(\tau^-\beta^-\rho^+)A(\delta^-\tau^-\rho^-)\Delta(\alpha\tau\rho)\delta_{\alpha\gamma}\\ &\quad-\sum_{\rho}A(\alpha^+\gamma^+\rho^+)A(\delta^-\beta^-\rho^-)\Delta(\gamma\beta\rho)\\ &\quad-\sum_{\rho}A(\delta^-\beta^-\rho^+)A(\alpha^+\gamma^+\rho^-)\Delta(\alpha\delta\rho)\Big]\mathcal{S}^{-1}(\gamma\delta)\,,\end{aligned} \quad (251)$$

where the first two terms on the right hand side are unconnected and represent the quark and antiquark self-energies, while the remaining two terms precisely give the t-channel one-gluon exchange kernel. In view of the expression in (80) and the definitions in (67) and (71), we can write

$$\begin{aligned}A(\alpha^+\gamma^+\rho^\pm)=&\,\mathrm{i}g(2\pi)^3\delta^3(\mathbf{p}_1-\mathbf{q}_1\mp\mathbf{k})\overline{u}_{\sigma_1}(\mathbf{p}_1)T^c\gamma_\mu\\ &\times u_{\sigma_1'}(\mathbf{q}_1)(2\pi)^{-3/2}(2\omega(\mathbf{k}))^{-1/2}\varepsilon^\lambda_\mu(\mathbf{k})\end{aligned} \quad (252)$$

and

$$\begin{aligned}A(\delta^-\beta^-\rho^\mp)=&\,\mathrm{i}g(2\pi)^3\delta^3(\mathbf{q}_2-\mathbf{p}_2\pm\mathbf{k})\overline{v}_{\sigma_2'}(\mathbf{q}_2)T^c\gamma_\mu\\ &\times v_{\sigma_2}(\mathbf{p}_2)(2\pi)^{-3/2}(2\omega(\mathbf{k}))^{-1/2}\varepsilon^\lambda_\mu(\mathbf{k})\\ =&-\mathrm{i}g(2\pi)^3\delta^3(\mathbf{q}_2-\mathbf{p}_2\pm\mathbf{k})\overline{u}_{\sigma_2}(\mathbf{p_2})(-T^{c*})\gamma_\nu\\ &\times u_{\sigma_2'}(\mathbf{q}_2)(2\pi)^{-3/2}(2\omega(\mathbf{k}))^{-1/2}\varepsilon^{\lambda'}_\nu(\mathbf{k})\,,\end{aligned} \quad (253)$$

where the last equality in (253) is obtained by the charge conjugation transformation. In the above, we have set $\mathbf{k}_\alpha=\mathbf{p}_1$, $\mathbf{k}_\beta=\mathbf{p}_2$, $\mathbf{k}_\gamma=\mathbf{q}_1$, $\mathbf{k}_\delta=\mathbf{q}_2$ and $\mathbf{k}_\rho=\mathbf{k}$. From (249), (250), (124) and (140), it is easy to get

$$\begin{aligned}&\Delta(\alpha\delta\rho)\\ &=\frac{1}{2}\int_{-\beta}^{\beta}\mathrm{d}\tau\mathrm{e}^{\mathrm{i}\omega_n\tau}\Delta_q(\mathbf{p}_1,\tau)\Delta_q(\mathbf{q}_2,\tau)\Delta_g(\mathbf{k},\tau)\\ &=\overline{n}^\alpha_f\overline{n}^\delta_f\overline{n}^\rho_b\frac{\mathrm{e}^{-\beta(\varepsilon_\alpha+\varepsilon_\delta+\omega_\rho)}-1}{\mathrm{i}\omega_n-\varepsilon_\alpha-\varepsilon_\delta-\omega_\rho}-\overline{n}^\alpha_f\overline{n}^\delta_f n^\rho_b\frac{\mathrm{e}^{-\beta(\varepsilon_\alpha+\varepsilon_\delta-\omega_\rho)}-1}{\mathrm{i}\omega_n-\varepsilon_\alpha-\varepsilon_\delta+\omega_\rho}\\ &\quad-\overline{n}^\alpha_f n^\delta_f\overline{n}^\rho_b\frac{\mathrm{e}^{-\beta(\varepsilon_\alpha-\varepsilon_\delta+\omega_\rho)}-1}{\mathrm{i}\omega_n-\varepsilon_\alpha+\varepsilon_\delta-\omega_\rho}+\overline{n}^\alpha_f n^\delta_f n^\rho_b\frac{\mathrm{e}^{-\beta(\varepsilon_\alpha-\varepsilon_\delta-\omega_\rho)}-1}{\mathrm{i}\omega_n-\varepsilon_\alpha+\varepsilon_\delta+\omega_\rho}\\ &\quad-n^\alpha_f\overline{n}^\delta_f\overline{n}^\rho_b\frac{\mathrm{e}^{\beta(\varepsilon_\alpha-\varepsilon_\delta-\omega_\rho)}-1}{\mathrm{i}\omega_n+\varepsilon_\alpha-\varepsilon_\delta-\omega_\rho}+n^\alpha_f\overline{n}^\delta_f n^\rho_b\frac{\mathrm{e}^{\beta(\varepsilon_\alpha-\varepsilon_\delta+\omega_\rho)}-1}{\mathrm{i}\omega_n+\varepsilon_\alpha-\varepsilon_\delta+\omega_\rho}\\ &\quad+n^\alpha_f n^\delta_f\overline{n}^\rho_b\frac{\mathrm{e}^{\beta(\varepsilon_\alpha+\varepsilon_\delta-\omega_\rho)}-1}{\mathrm{i}\omega_n+\varepsilon_\alpha+\varepsilon_\delta-\omega_\rho}-n^\alpha_f n^\delta_f n^\rho_b\frac{\mathrm{e}^{\beta(\varepsilon_\alpha+\varepsilon_\delta+\omega_\rho)}-1}{\mathrm{i}\omega_n+\varepsilon_\alpha+\varepsilon_\delta+\omega_\rho}\\ &\equiv\Delta\left(\mathbf{p_1},\mathbf{q_2},\mathbf{k};\omega_n\right)\end{aligned} \quad (254)$$

and

$$
\begin{aligned}
&\Delta(\gamma\beta\rho)\\
&=\frac{1}{2}\int_{-\beta}^{\beta}\mathrm{d}\tau\mathrm{e}^{\mathrm{i}\omega_n\tau}\Delta_q(\mathbf{p}_2,\tau)\Delta_q(\mathbf{q}_1,\tau)\Delta_g(\mathbf{k},\tau)\\
&=\overline{n}_f^{\gamma}\overline{n}_f^{\beta}\overline{n}_b^{\rho}\frac{\mathrm{e}^{-\beta(\varepsilon_\gamma+\varepsilon_\beta+\omega_\rho)}-1}{\mathrm{i}\omega_n-\varepsilon_\gamma-\varepsilon_\beta-\omega_\rho}-\overline{n}_f^{\gamma}\overline{n}_f^{\beta}n_b^{\rho}\frac{\mathrm{e}^{-\beta(\varepsilon_\gamma+\varepsilon_\beta-\omega_\rho)}-1}{\mathrm{i}\omega_n-\varepsilon_\gamma-\varepsilon_\beta+\omega_\rho}\\
&-\overline{n}_f^{\gamma}n_f^{\beta}\overline{n}_b^{\rho}\frac{\mathrm{e}^{-\beta(\varepsilon_\gamma-\varepsilon_\beta+\omega_\rho)}-1}{\mathrm{i}\omega_n-\varepsilon_\gamma+\varepsilon_\beta-\omega_\rho}+\overline{n}_f^{\gamma}n_f^{\beta}n_b^{\rho}\frac{\mathrm{e}^{-\beta(\varepsilon_\gamma-\varepsilon_\beta-\omega_\rho)}-1}{\mathrm{i}\omega_n-\varepsilon_\gamma+\varepsilon_\beta+\omega_\rho}\\
&-n_f^{\gamma}\overline{n}_f^{\beta}\overline{n}_b^{\rho}\frac{\mathrm{e}^{\beta(\varepsilon_\gamma-\varepsilon_\beta-\omega_\rho)}-1}{\mathrm{i}\omega_n+\varepsilon_\gamma-\varepsilon_\beta-\omega_\rho}+n_f^{\gamma}\overline{n}_f^{\beta}n_b^{\rho}\frac{\mathrm{e}^{\beta(\varepsilon_\gamma-\varepsilon_\beta+\omega_\rho)}-1}{\mathrm{i}\omega_n+\varepsilon_\gamma-\varepsilon_\beta+\omega_\rho}\\
&+n_f^{\gamma}n_f^{\beta}\overline{n}_b^{\rho}\frac{\mathrm{e}^{\beta(\varepsilon_\gamma+\varepsilon_\beta-\omega_\rho)}-1}{\mathrm{i}\omega_n+\varepsilon_\gamma+\varepsilon_\beta-\omega_\rho}-n_f^{\gamma}n_f^{\beta}n_b^{\rho}\frac{\mathrm{e}^{\beta(\varepsilon_\gamma+\varepsilon_\beta+\omega_\rho)}-1}{\mathrm{i}\omega_n+\varepsilon_\gamma+\varepsilon_\beta+\omega_\rho}\\
&\equiv\Delta(\mathbf{p}_2,\mathbf{q}_1,\mathbf{k};\omega_n)\,, \qquad (255)
\end{aligned}
$$

where

$$
\varepsilon_\alpha=\sqrt{\mathbf{p}_1^2+m_1},\quad \varepsilon_\beta=\sqrt{\mathbf{p}_2^2+m_2},\quad \varepsilon_\gamma=\sqrt{\mathbf{q}_1^2+m_1},\quad \varepsilon_\delta=\sqrt{\mathbf{q}_2^2+m_2},\quad \omega_\rho=|\mathbf{k}|\,. \qquad (256)
$$

It is noted here that the chemical potential is not taken into account for the bound state. Other expressions for the functions $\Delta(\alpha\delta\rho)$ and $\Delta(\gamma\beta\rho)$ may be given by making use of the expansions presented in (A.10) and (A.19) in the appendix. Since the expressions contain infinite series, it might be not convenient for our purpose. Upon inserting (252)–(255) into the last two terms in (251) and noticing the definition in (72), after completing the integration over $\mathbf{k}$ and the summation over the polarization index, we obtain

$$
\begin{aligned}
K(\alpha^+\beta^-;\gamma^+\delta^-;E)&=K(\mathbf{p}_1,\mathbf{p}_2;\mathbf{q}_1,\mathbf{q}_2;E)\\
&=(2\pi)^{-3}\delta^3(\mathbf{p}_1+\mathbf{p}_2-\mathbf{q}_1-\mathbf{q}_2)\\
&\quad\times V(\mathbf{p}_1,\mathbf{p}_2;\mathbf{q}_1,\mathbf{q}_2;E)\,, \qquad (257)
\end{aligned}
$$

where the $\mathrm{i}\omega_n$ in (254) and (255) has been replaced by E, and

$$
\begin{aligned}
V(\mathbf{p}_1,\mathbf{p}_2;\mathbf{q}_1,\mathbf{q}_2;E)&=\overline{u}_{\sigma_1}(\mathbf{p}_1)T^c\gamma_\mu u_{\sigma_1'}(\mathbf{q}_1)\overline{u}_{\sigma_2}(\mathbf{p}_2)T^{c*}\gamma_\mu\\
&\quad\times u_{\sigma_2'}(\mathbf{q}_2)D(\mathbf{p}_1,\mathbf{p}_2;\mathbf{q}_1,\mathbf{q}_2) \qquad (258)
\end{aligned}
$$

in which

$$
\begin{aligned}
D(\mathbf{p}_1,\mathbf{p}_2;\mathbf{q}_1,\mathbf{q}_2)&=\frac{1}{8\omega(\mathbf{p}_1-\mathbf{q}_1)}[\Delta(\mathbf{p}_1,\mathbf{q}_2,\mathbf{p}_1-\mathbf{q}_1)\\
&\quad+\Delta(\mathbf{p}_2,\mathbf{q}_1,\mathbf{p}_1-\mathbf{q}_1)]\mathcal{S}^{-1}(\mathbf{q}_1,\mathbf{q}_2)\,. \qquad (259)
\end{aligned}
$$

With the kernel given above, the equation in (192) can be written as

$$
\begin{aligned}
&[E-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2)]\chi_{P\alpha}(\mathbf{p}_1,\mathbf{p}_2)\\
&=\int\mathrm{d}^3q_1\mathrm{d}^3q_2K(\mathbf{p}_1,\mathbf{p}_2;\mathbf{q}_1,\mathbf{q}_2)\chi_{P\alpha}(\mathbf{q}_1,\mathbf{q}_2)\,. \qquad (260)
\end{aligned}
$$

When we introduce the cluster momenta

$$
\begin{aligned}
&\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2,\quad \mathbf{Q}=\mathbf{q}_1+\mathbf{q}_2,\quad \mathbf{q}=\eta_2\mathbf{p}_1-\eta_1\mathbf{p}_2,\\
&\mathbf{k}=\eta_2\mathbf{q}_1-\eta_1\mathbf{q}_2,\quad \eta_1=\frac{m_1}{m_1+m_2},\quad \eta_2=\frac{m_2}{m_1+m_2}\,, \qquad (261)
\end{aligned}
$$

in the center of mass frame, (260) will be represented as

$$
[E-\varepsilon(\mathbf{p}_1)-\varepsilon(\mathbf{p}_2)]\chi_{P\alpha}(\mathbf{q})=\int\frac{\mathrm{d}^3k}{(2\pi)^3}V(\mathbf{P},\mathbf{q},\mathbf{k})\chi_{P\alpha}(\mathbf{k}). \qquad (262)
$$

As we know, the spinor function in (258) can be written as $u_\sigma(\mathbf{p})=U(\mathbf{p})\varphi_\sigma$ where φ_σ is the spin wave function and $U(\mathbf{p})$ is the ordinary Dirac spinor determined by the Dirac equation. The spin wave functions may be absorbed into the amplitude $\psi(\alpha\beta;E)$ appearing in (193). Thus, corresponding to the align in (262), the equation in (193) will be represented as

$$
[E-\varepsilon_1(\mathbf{q})-\varepsilon_2(\mathbf{q})]\psi_{P\alpha}(\mathbf{q})=\int\frac{\mathrm{d}^3k}{(2\pi)^3}\widehat{V}(\mathbf{P};\mathbf{q},\mathbf{k})\psi_{P\alpha}(\mathbf{k})\,, \qquad (263)
$$

where $\psi_{P\alpha}(\mathbf{q})$ stands for the color singlet wave function given in the Pauli spinor space, and

$$
\begin{aligned}
&\widehat{V}(\mathbf{P};\mathbf{q},\mathbf{k})\\
&=-\frac{4}{3}g^2\overline{U}(\mathbf{p}_1)\gamma_\mu U(\mathbf{q}_1)\overline{U}(\mathbf{p}_2)\gamma^\mu U(\mathbf{q}_2)D(\mathbf{P},\mathbf{q},\mathbf{k}) \qquad (264)
\end{aligned}
$$

represents the one-gluon exchange interaction Hamiltonian which formally is the same as given in the case of zero temperature [17]. In (264), we have recovered the Minkowski metric for the γ-matrix in order to compare with the ordinary zero-temperature result and considered that the expectation value of the $q\overline{q}$ color operator $T_1^a(-T_2^{a*})$ in the color singlet equals $-\frac{4}{3}$.

8 Concluding remarks

In this paper, there are two new achievements. One is that the path-integral formalism of the thermal QCD has been correctly established in the coherent-state representation. The expression of the QCD generating functional formulated in this paper not only gives an alternative quantization of the QCD, but also provides a general method for calculating the partition function, the thermal Green functions and thereby other statistical quantities of QCD in the coherent-state representation. In particular, the generating functional enables us to carry out analytical calculations without being concerned with its discretized form. As one has seen from Sect. 4, the analytical calculation of the zero-order generating functional is simpler and more direct than the previous calculations performed in the discretized form given either in the coherent-state representation or in the position space [18–22]. The coherent-state path-integral formalism corresponds to the operator formalism represented in terms of creation and annihilation operators. In comparison with the latter formalism, which was frequently applied in the many-body theory [21, 29], the coherent-state path-integral formalism has the prominent advantage that in calculations within this formalism the use of the operator commutators and the Wick theorem is completely avoided. Therefore, it is more convenient for practical applications. It should be noted that although the QCD generating functional is derived in the Feynman gauge, the result is exact. This is because QCD is a gauge-independent theory. As shown in Sect. 4, in the partition function derived in the Feynman gauge, the unphysical part of the partition function given by the unphysical degrees of freedom of gluons is completely cancelled out by the partition function arising from the ghost particles. Certainly, the generating functional formulated in the coherent-state representation can be established in arbitrary gauges. But, in this case, the gluon propagator would have a rather complicated form due to the fact that the longitudinal part of the propagator will involve the polarization vector. Another point we would like to mention is that to formulate the quantization of thermal QCD in the coherent-state representation, we limit ourself to work in the imaginary-time formalism. There is no doubt that the theory can also be described in the real-time formalism. We leave the discussion of this subject to the future.

The main achievement of this paper is the foundation of a rigorous three-dimensional equation for the $q\overline{q}$ bound states at finite temperature. Especially, the interaction kernel in this equation is given by a closed expression in the coherent-state representation. This kernel, as shown in (226), contains only a few types of Green functions with some definite coefficients. We also give the corresponding closed expression represented in the position space. As shown in (231)–(242), in this expression only a few types of Green functions and commutators are involved. Therefore, the kernel cannot only easily be calculated by the perturbation method but also is suitable for nonperturbative investigations by using the lattice gauge approach and some others. Since the kernel contains all the interactions taking place in the bound state, obviously, the kernel and the equation presented in the preceding sections are very suitable to study quark deconfinement at high temperature, which is nowadays an important theoretical problem in high energy physics. It is expected that an accurate nonperturbative calculation of the kernel could come up in the future so as to give the problem of quark deconfinement a definite solution.

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Appendix: Derivation of the generating functional represented in the position space

To confirm the correctness of the results derived in Sect. 4, in this appendix we plan to derive the familiar perturbative expansion of the thermal QCD generating functional represented in the position space from the corresponding one given in Sect. 4. For this purpose, we need to derive the generating functional represented in position space for the free system, which can be written as

$$Z^0[J] = Z_g^0\left[J_\mu^a\right] Z_q^0\left[I,\overline{I}\right] Z_c^0\left[K^a,\overline{K}^a\right], \tag{A.1}$$

where $Z_g^0[J_\mu^a]$, $Z_q^0[I,\overline{I}]$ and $Z_c^0[K^a,\overline{K}^a]$ are the position space generating functionals arising respectively from the free gluons, quarks and ghost particles, and J_μ^a, I, $\overline{I}$, K^a and $\overline{K}^a$ are the sources coupled to gluon, quark and ghost particle fields respectively. In order to write out the $Z_q^0[I,\overline{I}]$, $Z_g^0[J_\mu^a]$ and $Z_c^0[K^a,\overline{K}^a]$ from the generating functionals given in (119), (138) and (150), it is necessary to establish relations between the sources introduced in the position space and in the coherent-state representation. Let us separately discuss the functionals $Z_q^0[I,\overline{I}]$, $Z_g^0[J_\mu^a]$ and $Z_c^0[K^a,\overline{K}^a]$. First we focus our attention on the functional $Z_q^0[I,\overline{I}]$. Usually, the external source terms of fermions in the generating functional given in the position space are of the form $\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x[\overline{I}(\mathbf{x},\tau)\psi(\mathbf{x},\tau)+\overline{\psi}(\mathbf{x},\tau)I(\mathbf{x},\tau)]$ [22, 32]. Substituting in this expression the Fourier expansions in (58) and (59) for the quark fields and for the sources in the following,

$$\begin{aligned} I(\mathbf{x},\tau) &= \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} I(\mathbf{k},\tau)\mathrm{e}^{\mathrm{i}\mathbf{kx}}, \\ \overline{I}(\mathbf{x},\tau) &= \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \overline{I}(\mathbf{k},\tau)\mathrm{e}^{-\mathrm{i}\mathbf{kx}}, \end{aligned} \tag{A.2}$$

we have

$$\begin{aligned} &\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[\overline{I}(\mathbf{x},\tau)\psi(\mathbf{x},\tau)+\overline{\psi}(\mathbf{x},\tau)I(\mathbf{x},\tau)\right] \\ &= \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3k \left[\eta_s^*(\mathbf{k},\tau)b_s(\mathbf{k},\tau)+b_s^*(\mathbf{k},\tau)\eta_s(\mathbf{k},\tau)\right. \\ &\quad \left.+\overline{\eta}_s^*(\mathbf{k},\tau)d_s(\mathbf{k},\tau)+d_s^*(\mathbf{k},\tau)\overline{\eta}_s(\mathbf{k},\tau)\right], \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} &\eta_s(,\tau) = \overline{u}_s(\mathbf{k})I(\mathbf{k},\tau), \quad \eta_s^*(\mathbf{k},\tau) = \overline{I}(\mathbf{k},\tau)u_s(\mathbf{k}), \\ &\overline{\eta}_s(\mathbf{k},\tau) = -\overline{I}(-\mathbf{k},\tau)v_s(\mathbf{k}), \quad \overline{\eta}_s^*(\mathbf{k},\tau) = -\overline{v}_s(\mathbf{k})I(-\mathbf{k},\tau). \end{aligned} \tag{A.4}$$

From these relations and the property of Dirac spinors, the relation in (137) is easily proved. On substituting (A.4) into (138), one can get

$$\begin{aligned} Z_q^0[I,\overline{I}] = Z_q^0 \exp\Bigg\{ &\int_0^\beta \mathrm{d}\tau_1 \int_0^\beta \mathrm{d}\tau_2 \int \mathrm{d}^3k\overline{I}(\mathbf{k},\tau_1) \\ &\times S_{\mathrm{F}}(\mathbf{k},\tau_1-\tau_2)I(\mathbf{k},\tau_2)\Bigg\}, \end{aligned} \tag{A.5}$$

where

$$S_{\mathrm{F}}(\mathbf{k},\tau_1-\tau_2)=[(\mathbf{k}+m)/\varepsilon(\mathbf{k})]\Delta_q(\mathbf{k},\tau_1-\tau_2) \quad (A.6)$$

with $\mathbf{k}=\gamma_\mu k^\mu$ and $k^\mu=(\mathbf{k},\mathrm{i}\varepsilon_n)$. By making use of the inverse transformation of (A.2), the generating functional in (A.5) is finally represented as [22, 32]

$$Z_q^0[I,\overline{I}] = Z_q^0\exp\left\{\int_0^\beta \mathrm{d}^4x_1\int_0^\beta \mathrm{d}^4x_2\overline{I}(x_1)S_{\mathrm{F}}(x_1-x_2)I(x_2)\right\}, \quad (A.7)$$

where $x=(\mathbf{x},\tau)$, $\mathrm{d}^4x=\mathrm{d}\tau\mathrm{d}^3x$ and

$$S_{\mathrm{F}}(x_1-x_2)=\int\frac{\mathrm{d}^3k}{(2\pi)^3}S_{\mathrm{F}}(\mathbf{k},\tau_1-\tau_2)\mathrm{e}^{\mathrm{i}\mathbf{kx}}. \quad (A.8)$$

It is well-known that the propagator $\Delta_q^{ss'}(\mathbf{k},\tau_1-\tau_2)$ is antiperiodic,

$$\begin{aligned}\Delta_q^{ss'}(\mathbf{k},\tau_1-\tau_2)&=-\Delta_q^{ss'}(\mathbf{k},\tau_1-\tau_2-\beta), \quad \text{if} \quad \tau_1\succ\tau_2\\ \Delta_q^{ss'}(\mathbf{k},\tau_1-\tau_2)&=-\Delta_q^{ss'}(\mathbf{k},\tau_1-\tau_2+\beta), \quad \text{if} \quad \tau_1\prec\tau_2\,.\end{aligned} \quad (A.9)$$

This can easily be proved from its representation in the operator formalism as shown in (46) or (154) with the help of the translation transformation $\widehat{b}_s(\tau)=\mathrm{e}^{\beta\widehat{K}}\,\widehat{b}_s\mathrm{e}^{-\beta\widehat{K}}$ and the trace property $\mathrm{Tr}(AB)=\mathrm{Tr}(BA)$. According to the antiperiodic property of the propagator, we have the following expansion:

$$\Delta_q(\mathbf{k},\tau)=\frac{1}{\beta}\sum_n\Delta_q(\mathbf{k},\varepsilon_n)\mathrm{e}^{-\mathrm{i}\varepsilon_n\tau}, \quad (A.10)$$

where $\tau=\tau_1-\tau_2$, $\varepsilon_n=\frac{\pi}{\beta}(2n+1)$, with n being an integer, and

$$\Delta_q(\mathbf{k},\varepsilon_n)=\int_0^\beta\mathrm{d}\tau\mathrm{e}^{\mathrm{i}\varepsilon_n\tau}\Delta_q(\mathbf{k},\tau)=\frac{\varepsilon(\mathbf{k})}{\varepsilon_n^2+\varepsilon(\mathbf{k})^2}\,. \quad (A.11)$$

Substituting (A.10) into (A.6) and noticing the above expression, it can be found that [22, 32]

$$S_{\mathrm{F}}(x_1-x_2)=\frac{1}{\beta}\sum_n\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}(\mathbf{x}_1-\mathbf{x}_2)-\mathrm{i}\varepsilon_n(\tau_1-\tau_2)}}{\boldsymbol{\gamma}\mathbf{k}+m-\mathrm{i}\varepsilon_n\gamma^0}, \quad (A.12)$$

which is the familiar expression for the thermal fermion propagator in the position space.

Next, we discuss the generating functional $Z_g^0[J_\mu^a]$. The source term of gluons in the generating functional given in the position space is commonly taken as $\int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3x\, J_\mu^a(\mathbf{x},\tau)A^{a\mu}(\mathbf{x},\tau)$ [22, 32]. Employing the expansions in (60) and in the following expression:

$$J_\mu^a(\mathbf{x},\tau)=\int\frac{\mathrm{d}^3k}{(2\pi)^{3/2}}J_\mu^a(\mathbf{k},\tau)\mathrm{e}^{\mathrm{i}\mathbf{kx}}, \quad (A.13)$$

we can write

$$\begin{aligned}&\int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3xJ_\mu^c(\mathbf{x},\tau)A^{c\mu}(\mathbf{x},\tau)\\ &=\int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3k\left[\xi_\lambda^{c*}(\mathbf{k},\tau)a_\lambda^c(\mathbf{k},\tau)+a_\lambda^{c*}(\mathbf{k},\tau)\xi_\lambda^c(\mathbf{k},\tau)\right],\end{aligned} \quad (A.14)$$

where

$$\xi_\lambda^c(\mathbf{k},\tau)=(2\omega(\mathbf{k}))^{-1/2}\epsilon_\lambda^\mu(\mathbf{k})J_\mu^c(\mathbf{k},\tau)=\xi_\lambda^{c*}(-\mathbf{k},\tau), \quad (A.15)$$

in which the last equality follows from $J_\mu^a(\mathbf{x},\tau)$ being a real function. Inserting the relations in (A.15) and then the inverse transformation of (A.13) into (119) and considering completeness of the polarization vectors, one may find the generating functional $Z_g^0[J_\mu^a]$ [22, 32],

$$Z_g^0[J_\mu^a]=Z_g^0\exp\left\{\frac{1}{2}\int_0^\beta\mathrm{d}^4x_1\int_0^\beta\mathrm{d}^4x_2\times J_\mu^a(x_1)D_{\mu\nu}^{ab}(x_1-x_2)J_\nu^b(x_2)\right\}, \quad (A.16)$$

where

$$D_{\mu\nu}^{ab}(x_1-x_2) =\delta^{ab}\delta_{\mu\nu}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\omega(\mathbf{k})}\Delta_g(\mathbf{k},\tau_1-\tau_2)\mathrm{e}^{\mathrm{i}\mathbf{k}(\mathbf{x}_1-\mathbf{x}_2)}. \quad (A.17)$$

By the same argument as mentioned for (A.9), it can be proved that the gluon propagator $\Delta_g(\mathbf{k},\tau_1-\tau_2)$ is a periodic function

$$\begin{aligned}\Delta_g(\mathbf{k},\tau_1-\tau_2)&=\Delta_g(\mathbf{k},\tau_1-\tau_2-\beta), \quad \text{if} \quad \tau_1\succ\tau_2\,;\\ \Delta_g(\mathbf{k},\tau_1-\tau_2)&=\Delta_g(\mathbf{k},\tau_1-\tau_2+\beta), \quad \text{if} \quad \tau_1\prec\tau_2\,.\end{aligned} \quad (A.18)$$

Therefore, we have the expansion

$$\Delta_g(\mathbf{k},\tau)=\frac{1}{\beta}\sum_n\Delta_g(\mathbf{k},\omega_n)\mathrm{e}^{-\mathrm{i}\omega_n\tau}, \quad (A.19)$$

where $\omega_n=\frac{2\pi n}{\beta}$ and

$$\Delta_g(\mathbf{k},\omega_n)=\int_0^\beta\mathrm{d}\tau\mathrm{e}^{\mathrm{i}\omega_n\tau}\Delta_g(\mathbf{k},\tau)=\frac{\omega(\mathbf{k})}{\varepsilon_n^2+\omega(\mathbf{k})^2}. \quad (A.20)$$

Upon substituting (A.19) and (A.20) in (A.17), we arrive at [22, 32]

$$D_{\mu\nu}^{ab}(x_1 - x_2) = \delta^{ab} \frac{1}{\beta} \sum_n \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\delta_{\mu\nu}}{\varepsilon_n^2 + \omega(\mathbf{k})^2} \times \mathrm{e}^{\mathrm{i}\mathbf{k}(\mathbf{x}_1 - \mathbf{x}_2) - \mathrm{i}\omega_n(\tau_1 - \tau_2)}, \qquad (A.21)$$

which just is the gluon propagator given in the position space and in the Feynman gauge.

Finally, we turn to the generating functional $Z_c^0[K^a, \overline{K}^a]$. In accordance with the expansions in (62) and (63) for the ghost particle fields and those for the external sources:

$$K^a(\mathbf{x}, \tau) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} K^a(\mathbf{k}, \tau) \mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}},$$
$$\overline{K}^a(\mathbf{x}, \tau) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \overline{K}^a(\mathbf{k}, \tau) \mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{x}}, \qquad (A.22)$$

the relation between the sources in the position space and in the coherent-state representation can be found to be

$$\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3 x \left[\overline{K}^a(\mathbf{x}, \tau) C^a(\mathbf{x}, \tau) + \overline{C}^a(\mathbf{x}, \tau) K(\mathbf{x}, \tau) \right]$$
$$= \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3 k [\zeta_a^*(\mathbf{k}, \tau) c_a(\mathbf{k}, \tau) + c_a^*(\mathbf{k}, \tau) \zeta_a(\mathbf{k}, \tau) + \overline{\zeta}_a^*(\mathbf{k}, \tau) \overline{c}_a(\mathbf{k}, \tau) + \overline{c}_a^*(\mathbf{k}, \tau) \overline{\zeta}_a(\mathbf{k}, \tau)], \qquad (A.23)$$

where

$$\zeta_a(\mathbf{k}, \tau) = (2\omega(\mathbf{k}))^{-1/2} K^a(\mathbf{k}, \tau),$$
$$\zeta_a^*(\mathbf{k}, \tau) = (2\omega(\mathbf{k}))^{-1/2} \overline{K}^a(\mathbf{k}, \tau),$$
$$\overline{\zeta}_a(\mathbf{k}, \tau) = -(2\omega(\mathbf{k}))^{-1/2} \overline{K}^a(-\mathbf{k}, \tau),$$
$$\overline{\zeta}_a^*(\mathbf{k}, \tau) = -(2\omega(\mathbf{k}))^{-1/2} K^a(-\mathbf{k}, \tau), \qquad (A.24)$$

from which the relations in (149) directly follow. When the above relations and the inverse transformations of (A.22) are inserted into (150), one can get

$$Z_c^0[K^a, \overline{K}^a] = Z_c^0 \exp\left\{ -\int_0^\beta \mathrm{d}^4 x_1 \int_0^\beta \mathrm{d}^4 x_2 \overline{K}^a(x_1) \times \Delta_c^{ab}(x_1 - x_2) K^b(x_2) \right\}, \qquad (A.25)$$

where

$$\Delta_c^{ab}(x_1 - x_2) = \delta^{ab} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\omega(\mathbf{k})} \Delta_g(\mathbf{k}, \tau_1 - \tau_2) \mathrm{e}^{\mathrm{i}\mathbf{k}(\mathbf{x}_1 - \mathbf{x}_2)}$$
$$= \delta^{ab} \frac{1}{\beta} \sum_n \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\varepsilon_n^2 + \omega(\mathbf{k})^2} \mathrm{e}^{\mathrm{i}\mathbf{k}(\mathbf{x}_1 - \mathbf{x}_2) - \mathrm{i}\omega_n(\tau_1 - \tau_2)}, \qquad (A.26)$$

which just is the free ghost particle propagator given in the position space [22, 32]. In the last equality of (A.26), the expansion given in (A.19) and (A.20) have been used.

With the generating functionals given in (A.7), (A.16) and (A.25), the zeroth-order generating functional in (A.1) is explicitly represented in terms of the propagators and external sources. Clearly, the exact generating functional can immediately be written from (95) as shown in the following:

$$Z[J] = \exp\left\{ -\int_0^\beta \mathrm{d}^4 x \mathcal{H}_\mathrm{I}\left(\frac{\delta}{\delta J(x)} \right) \right\} Z^0[J], \qquad (A.27)$$

where J stands for $I, \overline{I}, J_\mu^a$ K^a and $\overline{K}^a$, $\frac{\delta}{\delta J(x)}$ represents the differentials $\frac{\delta}{\delta \overline{I}(x)}$, $-\frac{\delta}{\delta I(x)}$, $\frac{\delta}{\delta J_\mu^a(x)}$, $\frac{\delta}{\delta \overline{K}^a(x)}$ and $-\frac{\delta}{\delta K^a(x)}$ and $\mathcal{H}_\mathrm{I}(\frac{\delta}{\delta J(x)})$ can be written from (57) when the field functions in (57) are replaced by the differentials with respect to the corresponding sources.

References

1. M. Gyulassy, Nucl. Phys. A **685**, 432 (2001)
2. D. Hardtke, in: Proceedings of XXI International Symposium on Lepton and Photon Interactions at High Energies, p. 329, at Fermilab (2003)
3. R. Stock, J. Phys. G: Nucl. Part. Phys. **30**, 633 (2004)
4. Y. Iwasaki, K. Kanaya, S. Kaya, S. Sakai, T. Yoshie, Phys. Rev. D **69**, 014 507 (2004)
5. M.D. Elia, A. Di Giacomo, E. Meggiolaro, Phys. Rev. D, 114 504 (2003)
6. A. Mòcsy, F. Sannino, K. Tuominen, Phys. Rev. Lett. **92**, 182 302 (2004)
7. Z. Yang, P. Zhuang, Phys. Rev. C **69**, 035 203 (2004)
8. H. Gies, Phys. Rev. D **63**, 025 013 (2001)
9. M. Asakawa, U. Heinz, B. Müller, Phys. Rev. Lett. **85**, 2072 (2000)
10. H. Li, C.M. Shakin, Phys. Rev. D **66**, 074 016 (2002)
11. S. Gupta, Phys. Rev. D **64**, 034 507 (2001)
12. S. Jesgarz, S. Lerma, P.O. Hess, O. Civitarese, M. Reboiro, Phys. Rev. C **67**, 055 210 (2003)
13. G.F. Burgio, M. Baldo, P.K. Sahu, H.-J. Schulze, Phys. Rev. C **66**, 025 802 (2002)
14. B. Sheikholeslami-Sabzevari, Phys. Rev. C **65**, 054 904 (2002)
15. S.S. Wu, J. Phys. G: Nucl. Part. Phys. **16**, 1447 (1990); Commun. Theor. Phys. **10**, 181 (1988)
16. J.C. Su, J.X. Chen, Phys. Rev. D **69**, 076 002 (2004)
17. J.C. Su, J. Phys. G: Nucl. Part. Phys. **30**, 1309 (2004)
18. A. Casher, D. Lurie, M. Revzen, J. Math. Phys. **9**, 1312 (1968)
19. L.S. Schulman, Techniques and Applications of Path Integration (Wiley-Interscience Publication, New York, 1981)
20. M. Beccaria, B. Allés, F. Farchioni, Phys. Rev. E **55**, 3870 (1997)
21. J.W. Negele, H. Orland, Quantum Many-Particle Systems, (Perseus Books Publishing, L.L. C, 1998)
22. M. Le Bellac, Thermal Field Theory (Cambridge University Press, 1996)
23. J.C. Su, Phys. Lett. A **268**, 279 (2000)
24. S.S. Schweber, J. Math, Phys. **3**, 831 (1962)

25. L.D. Faddeev, A.A. Slavnov, Gauge Fields: Introduction to Quantum Theory (The Benjamin/Cummings Publishing Company, Inc., 1980)
26. C. Itzykson, J.-B. Zuber, Quantum Field Theory (McGraw-Hill Inc., New York, 1980)
27. J.R. Klauder, Phys. Rev. D **19**, 2349 (1979)
28. R.J. Glauber, Phys. Rev. **131**, 2766 (1963)
29. A.L. Fetter, J.D. Walecka, Quantum Theory of Many-Particle Systems (McGraw-Hill, 1971)
30. D. Lurie, Particles and Fields (Interscience Publishers, New York, 1968)
31. H. Lehmann, Nuovo Cimento **11**, 342 (1954)
32. J.I. Kapusta, Finite-Temperature Field Theory (Cambridge University Press, 1989)